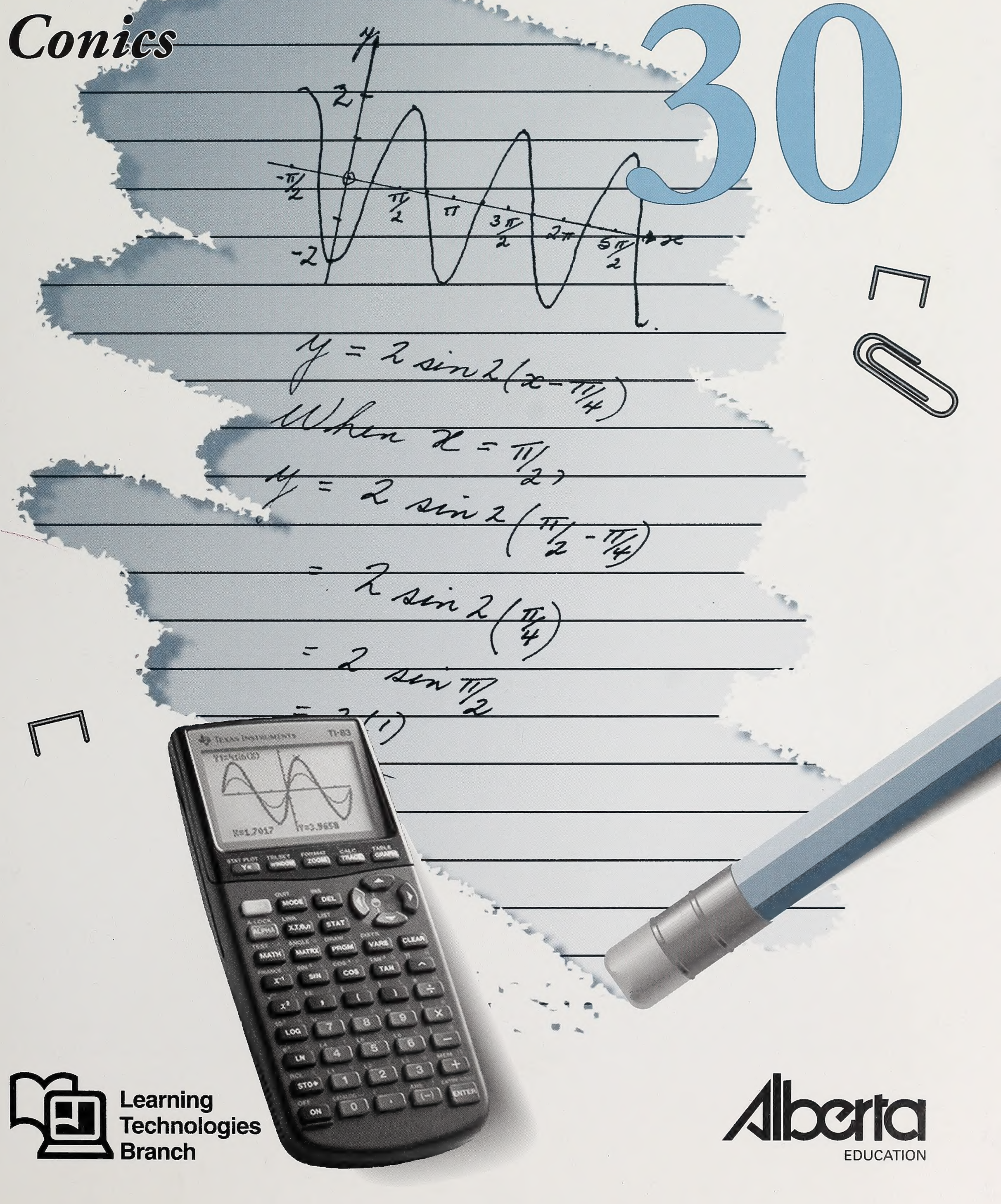
AL.2.2008-206
c.2
v.4

University of Alberta Library
0 1620 3452403 1

Module 4

MATHEMATICS 30

Conics



Learning Technologies Branch

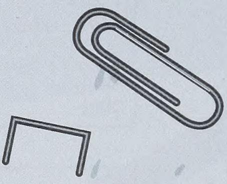
Alberta EDUCATION

Digitized by the Internet Archive
in 2016 with funding from
University of Alberta Libraries

https://archive.org/details/puremathematics304albe_1

Pure
MATHEMATICS
30

Module 4
Conics



Pure Mathematics 30
Module 4: Conics
Student Module Booklet
Learning Technologies Branch
ISBN 0-7741-2804-6

The Learning Technologies Branch acknowledges with appreciation the Alberta Distance Learning Centre and Pembina Hills Regional Division No. 7 for their review of this Student Module Booklet.

This document is intended for	
Students	✓
Teachers	✓
Administrators	
Home Instructors	
General Public	
Other	

You may find the following Internet sites useful:

- Alberta Education, **http://www.education.gov.ab.ca**
- Learning Technologies Branch, **http://www.education.gov.ab.ca/ltb**
- Learning Resources Centre, **http://www.lrc.education.gov.ab.ca**

Exploring the electronic information superhighway can be educational and entertaining. However, be aware that these computer networks are not censored. Students may unintentionally or purposely find articles on the Internet that may be offensive or inappropriate. As well, the sources of information are not always cited and the content may not be accurate. Therefore, students may wish to confirm facts with a second source.

Copyright © 2006, Alberta Education. This resource is owned by the Crown in Right of Alberta, as represented by the Minister of Education, Alberta Education, 10155 – 102 Street, Edmonton, Alberta, Canada T5J 4L5. All rights reserved.

No part of this courseware may be reproduced in any form, including photocopying (unless otherwise indicated), without the written permission of Alberta Education. This courseware was developed by or for Alberta Education. Third-party content has been identified by a © symbol and/or a credit to the source. Every effort has been made to acknowledge the original source and to comply with Canadian copyright law. If cases are identified where this effort has been unsuccessful, please notify Alberta Education so corrective action can be taken.

This courseware may contain one or more audio and/or multimedia components. Please review the Terms of Use Agreement on each for additional copyright information.

THIS COURSEWARE IS NOT SUBJECT TO THE TERMS OF A LICENCE FROM A COLLECTIVE OR LICENSING BODY, SUCH AS ACCESS COPYRIGHT.

UNIVERSITY LIBRARY
UNIVERSITY OF ALBERTA

Welcome

Pure Mathematics 30

Module 1
Transformations

Module 2
Exponents and Logarithms

Module 3
Sequences and Series

Module 4
Conics

Module 5
Trigonometry

Module 6
Combinatorics

Module 7
Probability and Statistics

Welcome to **Module 4**. We hope you'll enjoy your study of **Conics**.

Module 1 contains general information about the course components, additional resources, icons, assessment, and strategies for completing your work. If you do not have access to Module 1, contact your teacher to obtain this important information.

It is recommended that you work through the modules in the order given (from 1 to 7), since several concepts build on each other as you progress through the course.

CONTENTS

Module Overview

Assessment 2

Strategies for Completing a Module 3

Section 1

Circles and Ellipses

Activity 1: Conic Sections 6

Activity 2: The Circle 16

Activity 3: The Ellipse 26

Follow-up Activities 45

Extra Help 45

Enrichment 50

Conclusion 53

Assignment 53

Section 2

Hyperbolas and Parabolas

Activity 1: The Hyperbola 56

Activity 2: The Parabola 71

Follow-up Activities 81

Extra Help 81

Enrichment 85

Conclusion 88

Assignment 88

Module Summary

Final Module Assignment 89

Appendix

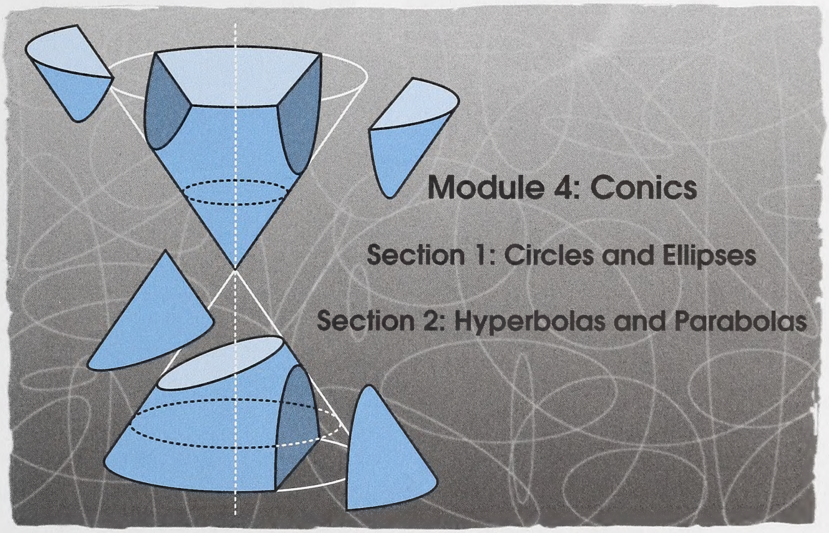
Glossary 92
Calculator Functions 93
Suggested Answers 94
Credits 169

Module Overview

Cross sections are useful in a variety of contexts, art and design, science and architecture, mathematics, engineering, and medicine. The study of cross sections can uncover interesting characteristics of ordinary objects that are otherwise hidden from view. The end of a culvert, for example, can be circular or elliptical depending on the angle at which it was cut.

Ancient Greek mathematicians were also interested in the geometry of everyday objects. From their observations, they generalized and developed an abstract approach to mathematics that has endured to the present day. For example, they knew that the circle, ellipse, parabola, and hyperbola can be obtained by slicing a cone. They contributed to the knowledge of these curves from this perspective.

In this module you will explore conics. In particular, you will investigate the circle, the ellipse, the hyperbola, and the parabola. Using your knowledge of transformations you gained in Module 1, you will examine the graphs and equations of these conics.


Module 4: Conics
Section 1: Circles and Ellipses
Section 2: Hyperbolas and Parabolas

Assessment

The document you are presently reading is called a Student Module Booklet. It will show you, step by step, what to do and how to do it.

This module, Conics, has several sections. Within each section, your work is grouped into activities. Within the activities, there are readings and questions for you to answer. By completing these questions you will construct your own learning, discover mathematical connections, and practise or apply what you have learned. The suggested answers in the Appendix of this Student Module Booklet will provide you with immediate feedback on your progress.

In this module you will be directed to the accompanying Assignment Booklets. You are expected to complete two Assignment Booklets for this module. Your grading in this module is based upon the assignments that you submit for assessment. The mark distribution is as follows:

Assignment Booklet 4A	
Section 1 Assignment	**45 marks**
Assignment Booklet 4B	
Section 2 Assignment	**40 marks**
Final Module Assignment	**15 marks**
TOTAL	**100 marks**

Strategies for Completing a Module

Before you begin, organize your materials: Student Module Booklet, textbook, Assignment Booklets, binder, lined paper, graph paper, graphing calculator, pens, pencils, and so on. Make sure you have a quiet area in which to work, away from distractions. Set up your time schedule.

To achieve success in this module, be sure to read all your instructions carefully and work slowly and systematically through the material. Remember, it's the work you do in this Student Module Booklet that will prepare you for your assignments and final test. Try to set realistic goals for yourself each day; and once you've set them, stick to them. Submit your assignments regularly, and don't forget to review your work before handing it in. Careful work habits will greatly increase your chances for success in Pure Mathematics 30.

Good luck!

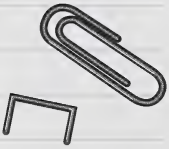
Section 1

Circles and Ellipses

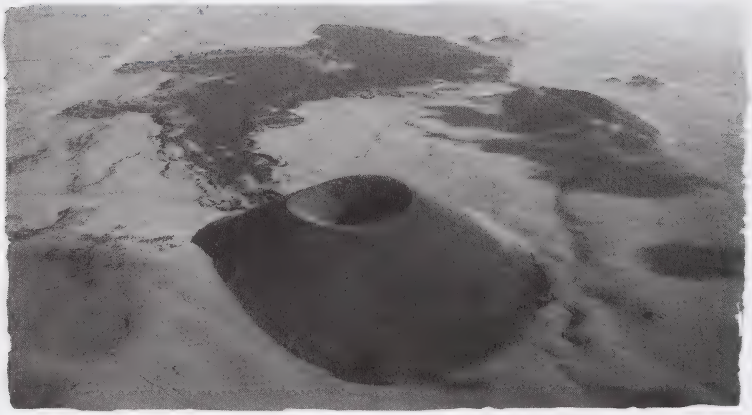
Can you pick out some of the planets in the solar system in the night sky? The word *planet* originates from the Greek word meaning “wanderer.” The ancient Greeks observed that the planets move (or wander) in relation to the fixed stars. As a matter of fact, the paths the planets follow around the Sun are ellipses. In the case of Earth and most planets, these ellipses are almost perfect circles. However, the orbits of Mercury and Pluto are noticeably elongated.

In his book on planetary motion (published in 1609), an astronomer and mathematician by the name of Johannes Kepler proposed that the orbit of each planet was elliptical. Before Kepler, astronomers thought the orbits of the planets were circles, because circles represented perfection and the eternal.

Did you know that both the circle and the ellipse can be obtained by slicing a cone? In this section you will first examine what curves or conic sections are possible when you slice a cone; then you will investigate the circle and the ellipse in detail. You will discover that the circle is a special case of the ellipse and that one can be changed into the other by applying transformations.



Activity 1: Conic Sections



TOM BEAN/CORBIS

The volcanic cinder cone in the photograph is one of nature's examples of a **right circular cone**. A right circular cone, as its name suggests, has a circular base and central axis that is perpendicular to the base. Other examples include the top of a sharpened pencil, an ice cream cone, and a conical pile of wheat in a field.

A Greek mathematician by the name of Apollonius of Perga (262 BC–190 BC), also known as "The Great Geometer," wrote an eight-book series called *Conics*. These books deal with **conic sections**—curves that can be obtained from the intersection of a plane and a right circular cone. One such conic is the circle; others include the ellipse, the hyperbola, and the parabola. These last three conics were introduced into mathematics by Apollonius of Perga.

For more information about Apollonius of Perga, visit the following website:

http://www-history.mcs.st-and.ac.uk/Biographies/Apollonius.html

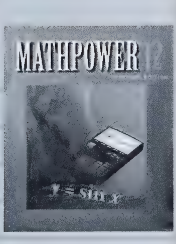
In this activity you will explore the intersection of both a plane and a conical surface and a plane and a cylindrical surface.

Mathematical Process

- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

right-circular cone: a cone with a circular base and a central axis that meets the base at right angles

conic section: a curve produced when a plane intersects a right-circular cone



Turn to page 134 of *MATHPOWER 12* and read "The Cone and Conic Sections"; then do the activity outlined in "Explore 1: Make a Model" and "Inquire."

1. Answer the following on pages 134 and 135 of the textbook.

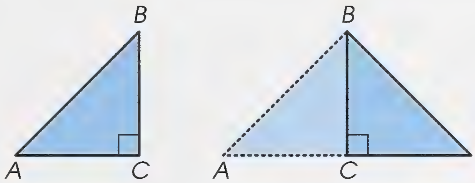
a. questions a. to d. of "Explore 2: Make a Model"
b. questions 1 to 3 of "Inquire"

Turn to page 94 to compare your responses with those in the Appendix.

In the preceding questions you explored **double-napped cones**. To obtain the lower nappe, you can rotate a right triangle about its altitude.

double-napped cone: two identical but opposite cones that share a common vertex

Look at right triangle *ABC*. Notice that a cone can be generated by revolving the triangle about side *BC*.

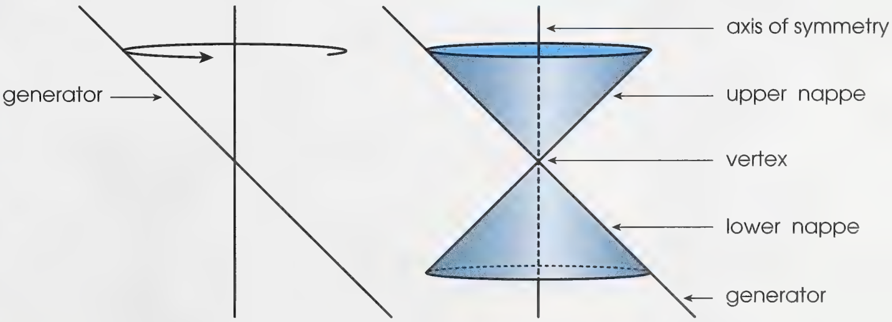


Single circular cone produced when ΔABC is rotated about side *BC*.

To obtain both nappes, simply extend *AB* and rotate as before. The line through *AB* is called the **generator** of the resulting double-napped, right-circular cone; and the line through *BC* is called the **central axis** (or axis of symmetry) of this cone.

generator: a line that sweeps out a surface, such as a cylinder or a right-circular cone, when rotated

central axis: the axis of symmetry of a right-circular cone

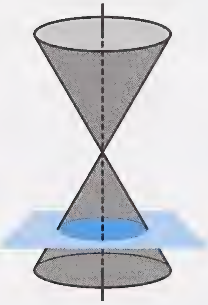


Remember, the nappes can be extended infinitely upward and downward.

From the Pure Mathematics 30 Companion CD, work through the segment *Primary Conic Sections* in the Computer-Assisted Instruction folder. This segment describes, in detail, how the circle, ellipse, parabola, and hyperbola arise from a plane intersecting a double-napped, right-circular cone.

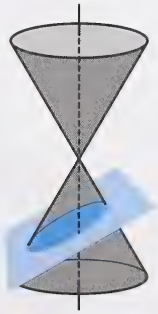
The following diagrams summarize the CD segment.

Circle

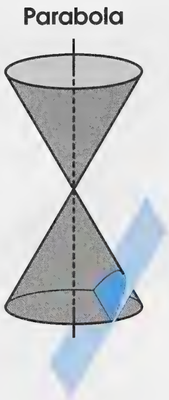


When a double-napped cone is intersected by a plane at a right angle to its axis, the cross section is a **circle**.

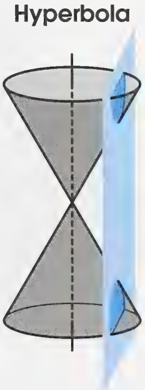
Ellipse



If the plane intersects one nappe of a double-napped cone at neither a right angle to the axis nor parallel to a generator, then the cross section is an **ellipse**.



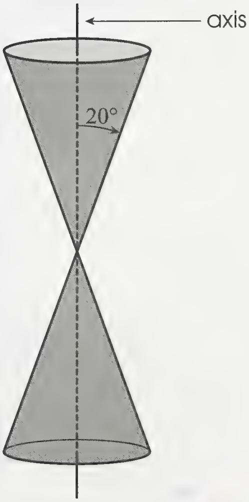
If the plane intersects the double-napped cone parallel to a generator, the cross section is a **parabola**.



When both nappes of a double-napped cone are intersected by a plane (not passing through the vertex), the cross section produces a **hyperbola**.

Now, it's your turn to practise these concepts.

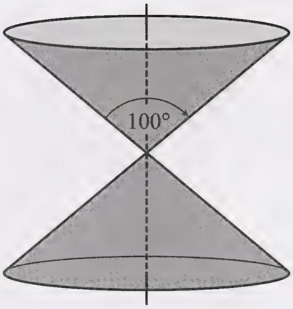
2. The angle between a generator of the given double-napped, right-circular cone and its central axis is 20°. At what angle or range of angles would an intersecting plane make with the central axis of this cone to produce each of the following conics?



a. a circle
b. a parabola
c. an ellipse
d. a hyperbola

3. The angle at the vertex of the double-napped, right-circular cone is 100°. At what angle or range of angles would an intersecting plane make with the central axis of this cone to produce each of the following conics?

   a. a circle
   b. a parabola
   c. an ellipse
   d. a hyperbola



Turn to page 95 to compare your responses with those in the Appendix.

You can now describe how to slice a double-napped cone to produce the standard conic sections.

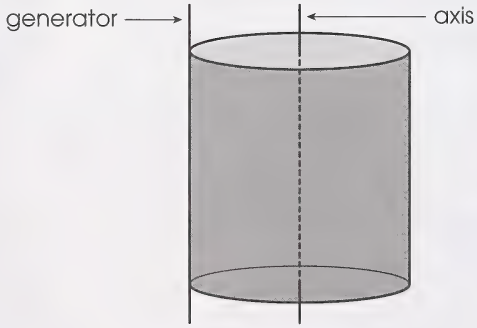
Some slices of a double-napped cone do not yield the usual circle, ellipse, parabola, and hyperbola. Before you can explore these sections in detail, you need to become familiar with the conics that can be obtained by slicing a cylinder.

Mathematical Process

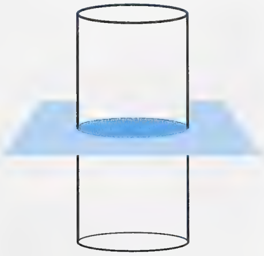
- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☒ Visualization

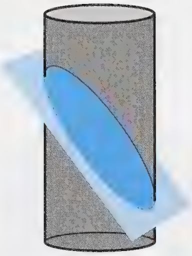
A cylinder can be thought of as a special case of the cone. Remember, the cone is formed by rotating a generator about the central axis. If you change the orientation of the generator so the angle between the generator and the central axis decreases until it is parallel to the central axis, the cone becomes a cylinder.

Cylinder



If a plane slices the cylinder at a right angle to the central axis (as for the cone), a **circle** is produced.

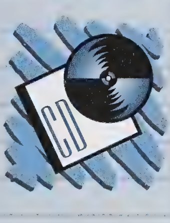




If the plane cuts the cylinder at any angle less than 90°, an **ellipse** is produced.

What do you think will happen if you keep tilting the cutting plane? Will the resulting curve always be an ellipse? To answer this question and to explore what happens when you slice the double-napped cone through its vertex, you need to know a little about the limiting cases of the familiar conics. In this module you will refer to these cases as **degenerate conic sections**.

degenerate conic section: a point, line, or pair of lines that arise as limiting forms of a conic



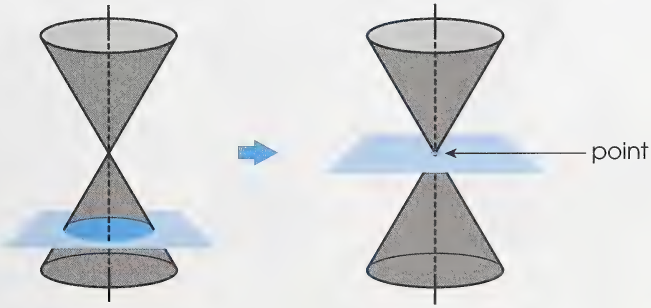
From the Pure Mathematics 30 Companion CD, work through the segment *Degenerate Conic Sections* in the Computer-Assisted Instruction folder.

The following diagrams summarize your findings.

Case 1

When a horizontal plane approaches the common vertex of a double-napped cone, the circle gets smaller. At the vertex, the circle becomes a **point**. You can say that the circle becomes a **point circle**.

point circle: a circle of radius zero

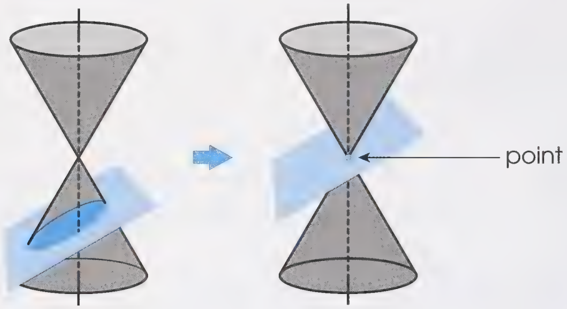


Mathematical Process

☐ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☐ Reasoning
☐ Technology
☒ Visualization

Case 2

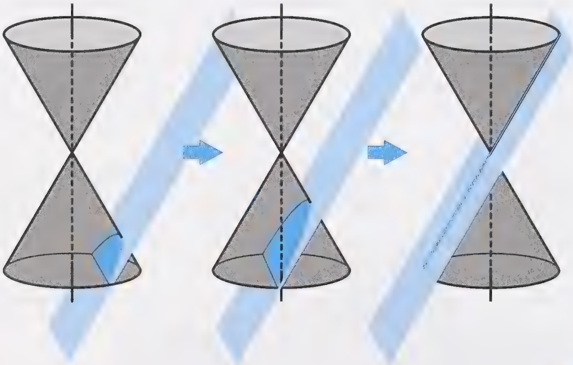
When the cutting plane is tilted at an angle between the horizontal and the generator, an ellipse is formed. When the cutting plane moves to the vertex, the ellipse becomes a point.



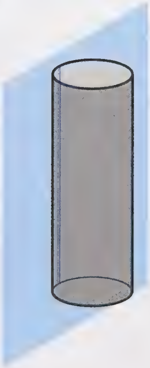
Case 3

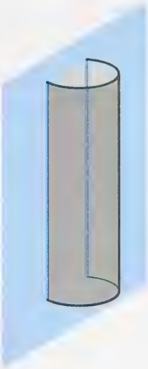
In this case the cutting plane is parallel to the generator of a double-napped cone or of a cylinder.

When a plane parallel to a generator approaches the common vertex of a double-napped cone, the parabola first gets larger and then becomes narrower. At the vertex, the parabola ultimately becomes a **straight line**.

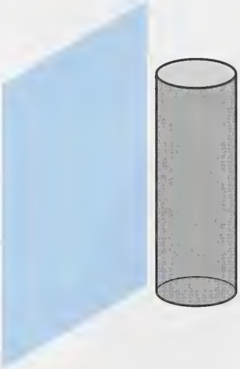


If the plane is parallel to the sides of the cylinder and it touches the cylinder, it will produce **one line**.





If the plane is parallel to the sides of the cylinder and cuts into the cylinder, it will produce **two parallel lines**.

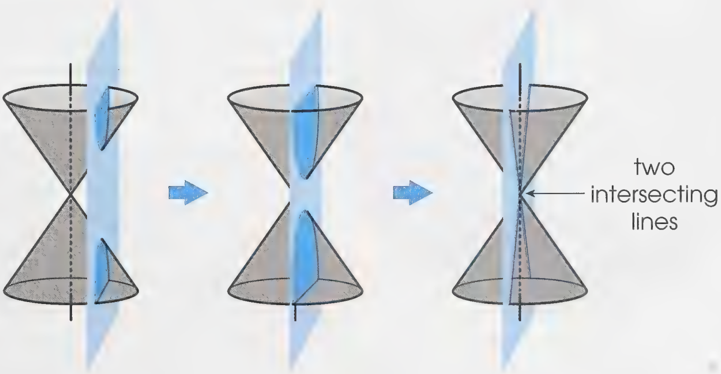


If the plane is parallel to the sides of the cylinder but doesn't touch the cylinder at all, it will produce **no curve**. This is referred to as the case of no locus or graph.

Case 4

In this case the cutting plane has been tilted beyond the angle of the generator, thus forming a hyperbola.

When the cutting plane approaches the vertex of the cone, the vertices of the hyperbola approach each other. At the vertex, the hyperbola becomes two straight lines crossing at the vertex.



4. Complete the following table by identifying the degenerate conics that correspond to the familiar conic sections.

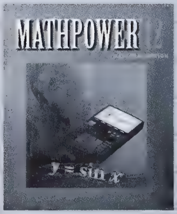
Conics	Degenerate Conic
Circle	
Ellipse	
Parabola	
Hyperbola	

Turn to page 96 to compare your responses with those in the Appendix.

Before you investigate the conic sections in detail, it's time to review and practise the main ideas in this activity.

5. Turn to page 135 of *MATHPOWER 12* and answer the following.

   a. questions 1 to 3 of "Practice"
   b. questions 4, 5, and 9 of "Applications and Problem Solving"



Turn to page 96 to compare your responses with those in the Appendix.

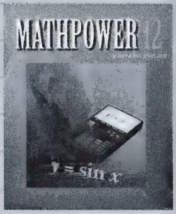
Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☒ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

Now Try This

One of the procedures you will be using in your analysis of conic sections is completing the square. For example, what constant needs to be added to $x^2 + 20x$ to make it a perfect square trinomial? Recall that the required value can be determined by dividing the coefficient of x by 2 and squaring the quotient. In this case the constant is $(20 \div 2)^2$ or 100. The resulting trinomial, $x^2 + 20x + 100$, is a perfect square and can be written in factored form as $(x + 10)^2$.

If you can complete the square mentally, you will save time in your written work! Here are some questions to practise your skills.



6. Turn to page 133 of *MATHPOWER 12* and answer questions 21 to 27 of "Mental Math."

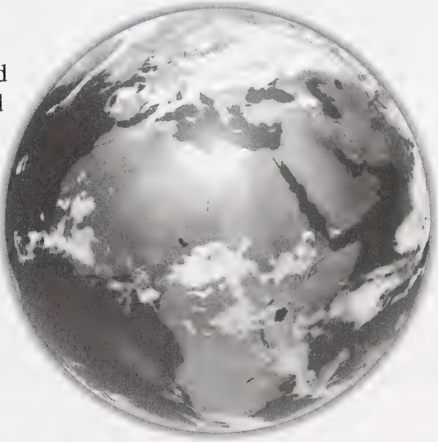
Turn to page 98 to compare your responses with those in the Appendix.

Mathematical Process

- ☒ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

Looking Back

In this activity you examined how the circle, ellipse, parabola, hyperbola, and their degenerate forms can be obtained from the intersection of a plane and a double-napped, right-circular cone or from the intersection of a plane and a cylinder.



In your journal, record as many examples of circles, ellipses, parabolas, and hyperbolas that you can think of that occur in nature. Compare your lists with those of other students taking Pure Mathematics 30.

Activity 2: The Circle

Mathematical Process

- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

Are you interested in night photography? Because of limited light, longer exposure times are necessary. However, with longer exposure times, special effects are possible. In the photograph, because Earth spins on its axis, the stars trace out circular arcs in the night sky. In the northern hemisphere, the common centre of these arcs is near Polaris, the North Star, which is less than 1° from the north celestial pole.

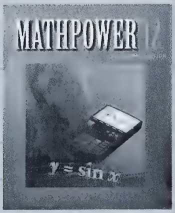
Did you know that the location of the north celestial pole changes because of Earth's precession (wobbling on its axis)? Just like the axis of a spinning top that traces out a **circle**, the axis of Earth slowly sweeps out a double-napped cone. As a result, Earth's poles trace out circles against the stars approximately every 26 000 years. About 4500 years ago, the north celestial pole pointed to a star named Thuban, the snake, in the constellation Draco (or dragon).

circle: the conic produced by slicing a right-circular cone at right angles to its central axis.

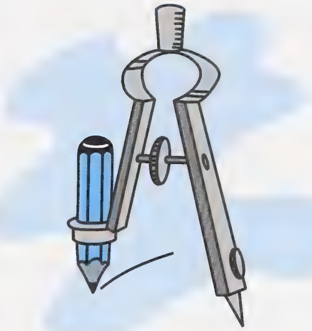
If you have access to the Internet, you can find more information about the precession of Earth at the following website:

http://csep10.phys.utk.edu/astr161/lect/time/precession.html

In this activity you will examine the circle—the first of the four conic sections you will explore in detail. You will investigate equations of circles centred at the origin and the transformations required to move it off the origin; and you will analyze circles from their equations.



Turn to page 138 of *MATHPOWER 12* and read "The Circle."

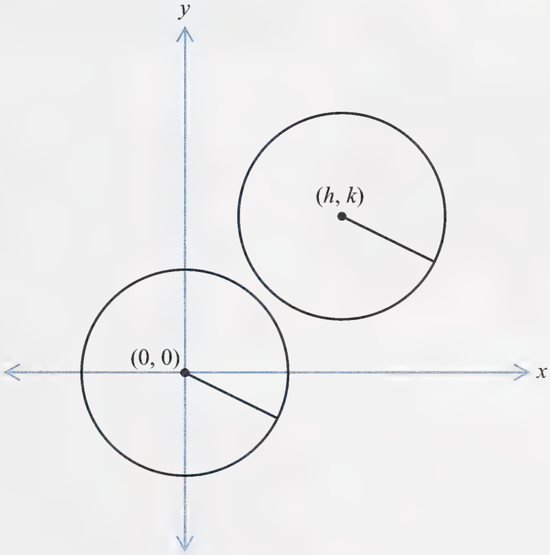


1. Answer the following on pages 138 and 139 of the textbook.

- **a.** "Explore 1: Use a Model"
- **b.** questions 1 to 4 of "Inquire"
- **c.** "Explore 2: Use a Model"
- **d.** questions 1, 2, and 4 of "Inquire"

Turn to page 99 to compare your responses with those in the Appendix.

By doing the previous questions, you determined that the standard form of the equation of a circle with radius r and centred at the origin is $x^2 + y^2 = r^2$. You then determined that the standard form of the equation of a circle with radius r and centred at point (h, k) is $(x - h)^2 + (y - k)^2 = r^2$.



This equation makes sense if you think in terms of transformations. Recall from Module 1 that the graph of $y - k = f(x - h)$ is the graph of $y = f(x)$ translated h units horizontally and k units vertically.

Mathematical Process

- ■ Communication
- ☒ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ☒ Reasoning
- ■ Technology
- ■ Visualization

Example

Determine the equation of a circle with radius 3 and centre

a. at the origin

b. (4, −2)

Solution

a. The standard form of a circle at the origin is $x^2 + y^2 = r^2$.

Substitute $r = 3$ into the equation.

$$\begin{aligned} x^2 + y^2 &= 3^2 \\ x^2 + y^2 &= 9 \end{aligned}$$

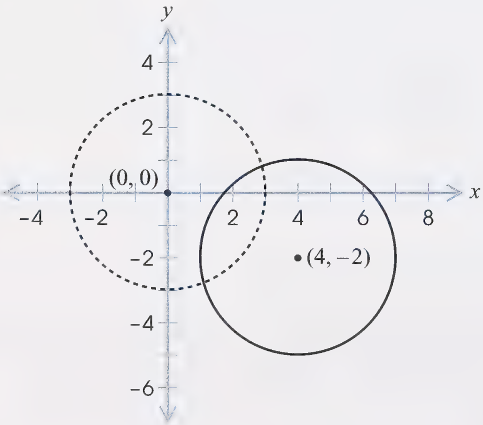
b. **Method 1: Using the Standard Form of a Circle**

The standard form of a circle centred at (h, k) is $(x - h)^2 + (y - k)^2 = r^2$.

Because $(h, k) = (4, -2)$ and the radius is 3, substitute $h = 4$, $k = -2$, and $r = 3$ into the equation.

$$\begin{aligned} (x-4)^2 + [y-(-2)]^2 &= 3^2 \\ (x-4)^2 + (y+2)^2 &= 9 \end{aligned}$$

Method 2: Using Transformations



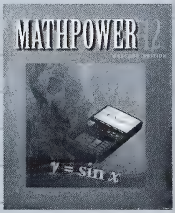
Because the centre of the circle is 4 units to the right and 2 units down from the origin, $h = 4$ and $k = -2$.

Substitute $(x - 4)$ for x, $[y-(-2)]$ for y, and 3 for r in the original equation.

$x - h = x - 4$
and
$y - k = y - (-2)$

$$x^2 + y^2 = r^2 \quad \leftarrow \text{original equation}$$
$$(x-4)^2 + [y-(-2)]^2 = 3^2 \quad \leftarrow \text{transformed equation}$$
$$(x-4)^2 + (y+2)^2 = 9$$

Now, you can find the equation of a circle two ways: using the standard form of a circle or using transformations.



Turn to page 139 of *MATHPOWER 12* and read from the red line near the top of the page to the red line near the top of page 140, working through Examples 1 and 2.

2. Answer the following on pages 141 and 142 of the textbook.

- **a.** questions 8, 15, 16, 27, 29, and 32 of "Practice"
- **b.** questions 55 and 56 of "Applications and Problem Solving"

Turn to page 102 to compare your responses with those in the Appendix.

In the preceding questions you worked with both the standard form, $(x - h)^2 + (y - k)^2 = r^2$, and the general form, $x^2 + y^2 + Dx + Ey + F = 0$, of a circle. As a matter of fact, the general form of any second-degree equation in two variables is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where at least one of A, B, and C is non-zero.

In Pure Mathematics 30 you will only consider equations for which $B = 0$. You will discover that the circle and other conics parallel to the *x*- and *y*-axes have equations that can be expressed in the following general form.

General Form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

Example

a. Write $(x-2)^2 + (y-3)^2 = 49$ in general form; and state the values of the coefficients *A*, *B*, *C*, *D*, *E*, and *F*.

b. How many different equations written in general form can represent the circle in part a?

Solution

a. Expand.

$$\begin{aligned}(x-2)^2 + (y-3)^2 &= 49\\ (x^2-4x+4)+(y^2-6y+9) &= 49\\ x^2+y^2-4x-6y+13 &= 49\\ x^2+y^2-4x-6y-36 &= 0\end{aligned}$$

The equation of the circle in general form is $x^2 + y^2 - 4x - 6y - 36 = 0$.

Compare $x^2 + y^2 - 4x - 6y - 36 = 0$ with $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Therefore, the coefficients are $A = 1$, $B = 0$, $C = 1$, $D = -4$, $E = -6$, and $F = -36$.

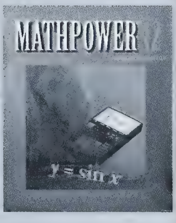
b. There are an infinite number of equations written in general form that represent the circle in part a. Simply multiply each term of the equation by any non-zero real number. For example, the following equations represent the same circle:

- $x^2 + y^2 - 4x - 6y - 36 = 0$ ← original equation
- $2x^2 + 2y^2 - 8x - 12y - 72 = 0$ ← Multiply each term by 2.
- $-3x^2 - 3y^2 + 12x + 18y + 108 = 0$ ← Multiply each term by –3.

In general, given an equation of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where $A = C \neq 0$, the equation represents a circle or one of its degenerate forms.

Now, suppose a circle is given in general form, how can you write it in standard form to determine its centre and radius?

Turn to page 140 of *MATHPOWER 12* and work through Example 3.



3. Answer questions 36, 37, and 39 of "Practice" on page 141 of the textbook.

Turn to page 106 to compare your responses with those in the Appendix.

In the preceding questions you found the centre and radius of circles from their equations written in general form. Next, you will use the centre and radius of a circle to determine its domain and range.

Example

Find the domain and range of the graph of $2x^2 + 2y^2 - 8x - 12y - 8 = 0$.

Solution

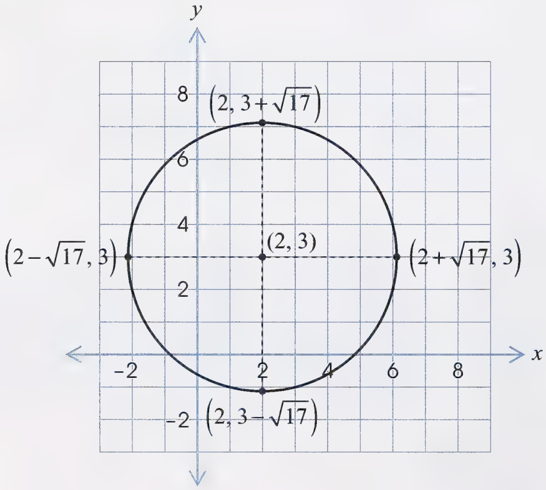
Because $A = C \neq 0$ and $B = 0$, the equation represents a circle or one of its degenerate forms.

Find the centre and the radius of the circle.

$$\begin{aligned} 2x^2+2y^2-8x-12y-8&=0 \\ x^2+y^2-4x-6y-4&=0 && \leftarrow \text{Divide each term by 2 so } A=C=1. \\ x^2-4x+y^2-6y&=4 && \leftarrow \text{Group the terms.} \\ \left[(x^2-4x+4)-4\right]+\left[(y^2-6y+9)-9\right]&=4 && \leftarrow \text{Complete the square.} \\ (x-2)^2-4+(y-3)^2-9&=4 \\ (x-2)^2+(y-3)^2&=17 \end{aligned}$$

The centre is at (2, 3), and the radius is $\sqrt{17}$.

Graph the circle to help determine the domain and range.



According to the graph, the domain (x-values) and range (y-values) can be determined by adding and subtracting $\sqrt{17}$ to the coordinates of the centre. Therefore, the domain is $2-\sqrt{17} \leq x \leq 2+\sqrt{17}$ and the range is $3-\sqrt{17} \leq y \leq 3+\sqrt{17}$.

Mathematical Process
- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ☒ Visualization

Example

Find the domain and range of the graph of $x^2 + y^2 - 4x - 6y + 13 = 0$.

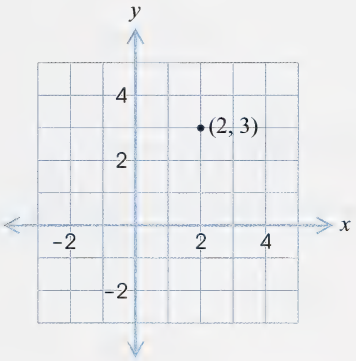
Solution

Because $A = C \neq 0$ and $B = 0$, the equation represents a circle or one of its degenerate forms.

Find the centre and the radius of the circle.

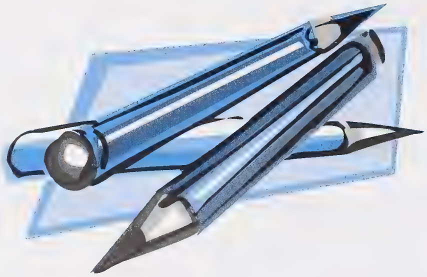
$$\begin{aligned} x^2 + y^2 - 4x - 6y + 13 &= 0 \\ x^2 - 4x + y^2 - 6y &= -13 \\ \left[(x^2 - 4x + 4) - 4\right] + \left[(y^2 - 6y + 9) - 9\right] &= -13 \\ (x-2)^2 - 4 + (y-3)^2 - 9 &= -13 \\ (x-2)^2 + (y-3)^2 &= 0 \end{aligned}$$

The centre is at (2, 3), and the radius is 0. This is an example of a point circle.



Therefore, the domain is $x = 2$ and the range is $y = 3$.

Example



Find the domain and range of the graph of $x^2 + y^2 - 4x - 6y + 17 = 0$.

Solution

Because $A = C \neq 0$ and $B = 0$, the equation represents a circle or one of its degenerate forms.

Find the centre and the radius of the circle.

$$\begin{aligned} x^2 + y^2 - 4x - 6y + 17 &= 0 \\ x^2 - 4x + y^2 - 6y &= -17 \\ \left[(x^2 - 4x + 4) - 4\right] + \left[(y^2 - 6y + 9) - 9\right] &= -17 \\ (x-2)^2 - 4 + (y-3)^2 - 9 &= -17 \\ (x-2)^2 + (y-3)^2 &= -4 \end{aligned}$$

Because $r = \sqrt{-4}$ is non-real, the circle does not exist. This is another form of a degenerate circle—one without a graph. Therefore, the domain and range are undefined.

4. Determine the domain and range of the following.

a. $x^2 + y^2 - 8x + 2y - 8 = 0$
b. $x^2 + y^2 + 12x + 16y = 0$
c. $x^2 + y^2 + x + 2y + 1.25 = 0$

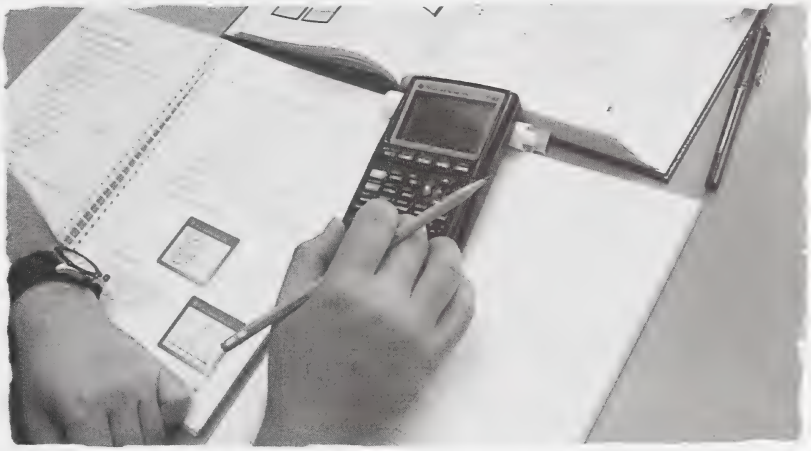
Turn to page 107 to compare your responses with those in the Appendix.

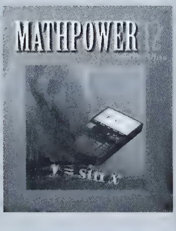
Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ■ Visualization

Now Try This

You can use your graphing calculator to analyse circles. However, most graphing calculators will not allow you to enter a relation such as $x^2 + y^2 + x + 2y = 0$ directly. If you want to use the equation, you must rewrite it in the form $y = f(x)$.





Turn to page 140 of *MATHPOWER 12* and work through Example 4.

5. Answer questions 40 and 45 of "Practice" on page 141 of the textbook.

Turn to page 109 to compare your responses with those in the Appendix.

Mathematical Process

- ☒ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

Looking Back

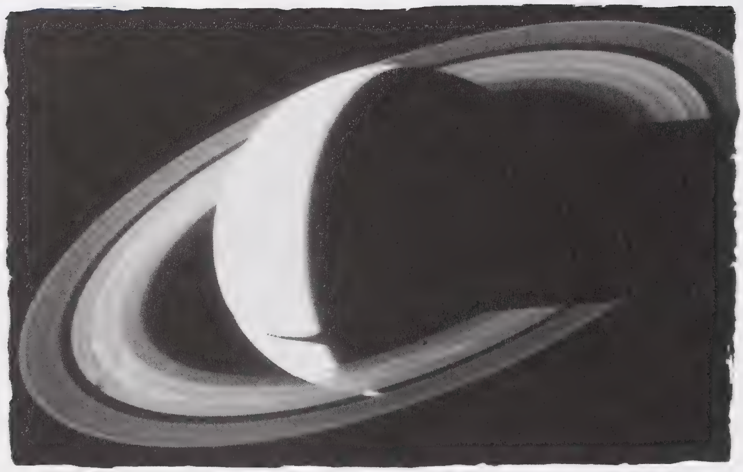
In this activity you first examined the equations of circles centred at the origin in the coordinate plane. You then translated these circles, centring them at any point, (h, k), in the coordinate plane and determined their new equations. You discovered that the equations of circles are usually written in one of two forms: standard form or general form. You converted equations between forms and used the standard form to identify the centre and radius of each circle. Finally, you determined the domain and range of a circular graph from its centre and radius.

Research the locus definition of a circle. Did you use the locus definition in this activity? What does the word *locus* mean in mathematics? Are there other examples of loci besides the circle. Summarize your finding in your journal.

Activity 3: The Ellipse

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ☒ Reasoning
- ■ Technology
- ☒ Visualization

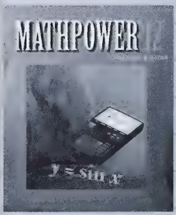


The rings of Saturn, which are actually circular, appear elliptical when the perspective of the camera changes. Painters use this principle in their art. The circular lip of a vase in a still life would be drawn as an **ellipse** in order to create the illusion that the picture is three-dimensional. If there is a glass or cup nearby, look at its top. Does it look like a circle or does it look like an ellipse?

ellipse: a conic produced by stretching or compressing a circle along an axis of symmetry

In this activity you will examine ellipses and their equations. You will discover that a circle is a special case of the ellipse and that ellipses can be formed by stretching or compressing circles. You will use this relationship to obtain the equations of ellipses by transforming the equations of circles.

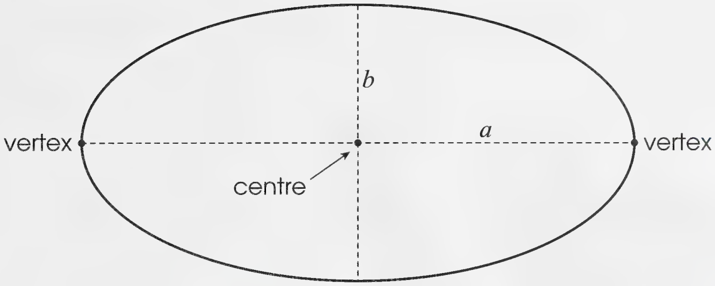
Turn to page 143 of *MATHPOWER 12* and read "The Ellipse."



1. Answer the following on page 143 of the textbook.

   a. "Explore: Use a Model"
   b. questions 1 to 7 of "Inquire"

Turn to page 111 to compare your responses with those in the Appendix.



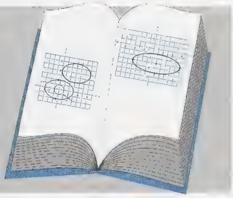
An ellipse has two **axes of symmetry**: a major axis and a minor axis. The **major axis** is the longer axis, and its length is 2 a. The endpoints of the major axis are called the **vertices of the ellipse**. The **minor axis** is the shorter axis, and its length is 2 b. These axes intersect at the **centre** of the ellipse.

axis of symmetry: a line in which the two halves of a curve reflect into each other

major axis: the longer axis of symmetry in an ellipse

minor axis: the shorter axis of symmetry in an ellipse

Note: In Pure Mathematics 30 you will study only those ellipses with horizontal and verical axes of symmetry. In future math courses you will rotate the axes and explore how a rotation affects the equations of an ellipse.



First, you will examine ellipses centred at the origin and with axes of symmetry along the coordinate axes.

Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☒ Technology
- ☐ Visualization

Example

Use your graphing calculator to graph the following ellipse; then determine the intercepts, the vertices, the endpoints of the minor axis, and the domain and range.

$$\frac{x^2}{64}+\frac{y^2}{25}=1$$

Solution

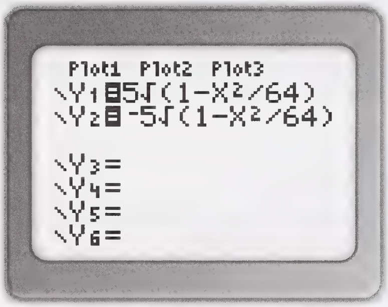
Use a square display to obtain a graph with minimal distortion by pressing ZOOM 5 (5:Zsquare).

First, solve the equation for y so you can enter this equation in the form $y = f(x)$.

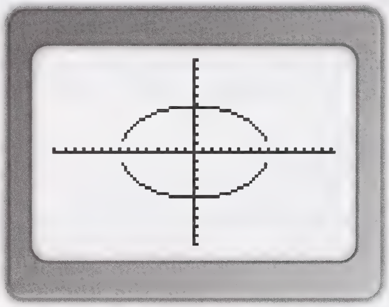
$$\begin{aligned}\frac{x^2}{64}+\frac{y^2}{25}&=1\\ \frac{y^2}{25}&=1-\frac{x^2}{64}\\ y^2&=25\left(1-\frac{x^2}{64}\right)\\ y&=\pm 5\sqrt{1-\frac{x^2}{64}}\end{aligned}$$

Now, enter $y = 5\sqrt{1-\frac{x^2}{64}}$ as Y_1 and $y = -5\sqrt{1-\frac{x^2}{64}}$ as Y_2.

Y= 5 2nd [√‾] 1 − X,T,Θ,n x^2 ÷ 6 4
) ENTER (−) 5 2nd [√‾] 1 − X,T,Θ,n x^2 ÷
6 4) GRAPH



```
Plot1 Plot2 Plot3
\Y1■5√(1-X²/64)
\Y2■-5√(1-X²/64)
\Y3=
\Y4=
\Y5=
\Y6=
```



You can use the Table feature on your calculator to confirm that the x-intercepts are ±8 and the y-intercepts are ±5, or you can find the intercepts algebraically.

To find the x-intercepts, let $y = 0$.

$$
\begin{aligned}
\therefore \frac{x^2}{64} + \frac{y^2}{25} &= 1 \\
\frac{x^2}{64} + \frac{0^2}{25} &= 1 \\
\frac{x^2}{64} &= 1 \\
x^2 &= 64 \\
x &= \pm 8
\end{aligned}
$$

To find the y-intercepts, let $x = 0$.

$$
\begin{aligned}
\therefore \frac{x^2}{64} + \frac{y^2}{25} &= 1 \\
\frac{0^2}{64} + \frac{y^2}{25} &= 1 \\
\frac{y^2}{25} &= 1 \\
y^2 &= 25 \\
y &= \pm 5
\end{aligned}
$$

The major axis lies along the x-axis; therefore, the ellipse's vertices (the endpoints of the major axis) are obtained from the x-intercepts. The vertices are $(8, 0)$ and $(-8, 0)$.

The minor axis lies along the y-axis; therefore, the endpoints of the minor axis are obtained from the y-intercepts. The endpoints of the minor axis are $(0, 5)$ and $(0, -5)$.

Using the intercepts, the domain is $-8 \leq x \leq 8$ and the range is $-5 \leq y \leq 5$.

Example

Use your graphing calculator to graph the following ellipse; then determine the intercepts, the vertices, the endpoints of the minor axis, and the domain and range.

$$\frac{x^2}{25} + \frac{y^2}{64} = 1$$

Mathematical Process

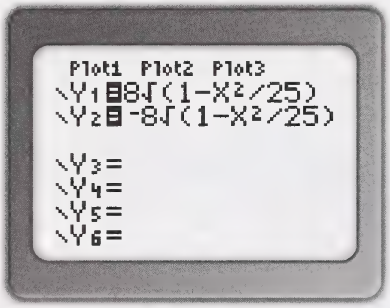
- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ■ Visualization

Solution

First, solve the equation for y so you can enter this equation in the form $y = f(x)$.

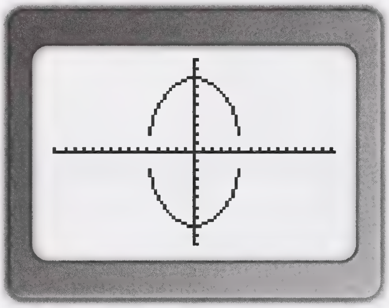
$$\begin{aligned} \frac{x^2}{25} + \frac{y^2}{64} &= 1 \\ \frac{y^2}{64} &= 1 - \frac{x^2}{25} \\ y^2 &= 64\left(1 - \frac{x^2}{25}\right) \\ y &= \pm 8\sqrt{1 - \frac{x^2}{25}} \end{aligned}$$

Now, enter $y = 8\sqrt{1 - \frac{x^2}{25}}$ as Y_1 and $y = -8\sqrt{1 - \frac{x^2}{25}}$ as Y_2.



```
Plot1 Plot2 Plot3
\Y1=8√(1-X²/25)
\Y2=-8√(1-X²/25)

\Y3=
\Y4=
\Y5=
\Y6=
```

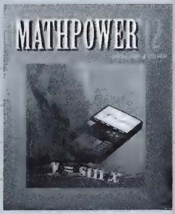


The x-intercepts are $(5, 0)$ and $(-5, 0)$, and the y-intercepts are $(0, 8)$ and $(0, -8)$.

The major axis lies along the y-axis; therefore, the ellipse's vertices (the endpoints of the major axis) are obtained from the y-intercepts. The vertices are $(0, 8)$ and $(0, -8)$.

The minor axis lies along the x-axis; therefore, the endpoints of the minor axis are obtained from the x-intercepts. The endpoints of the minor axis are $(5, 0)$ and $(-5, 0)$.

Using the intercepts, the domain is $-5 \le x \le 5$ and the range is $-8 \le y \le 8$.



Mathematical Process

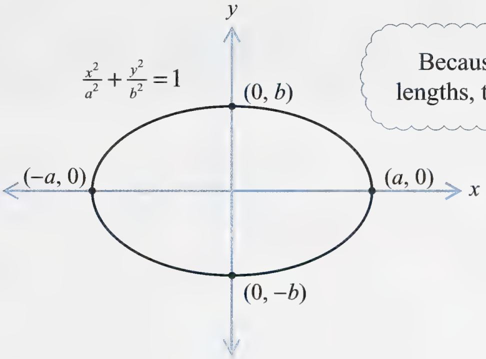
- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☒ Visualization

2. Turn to page 150 of *MATHPOWER 12* and answer questions 11 and 12 of "Practice."

Turn to page 112 to compare your responses with those in the Appendix.

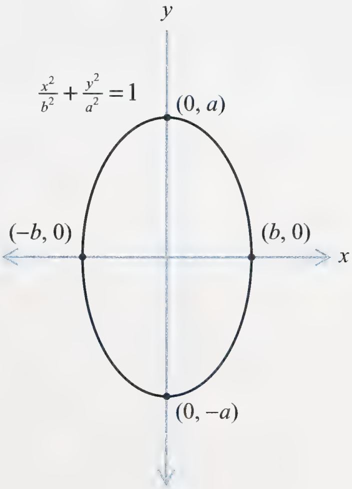
After working through the preceding examples and answering question 2, you can develop the **standard form of an ellipse** centred at the origin.

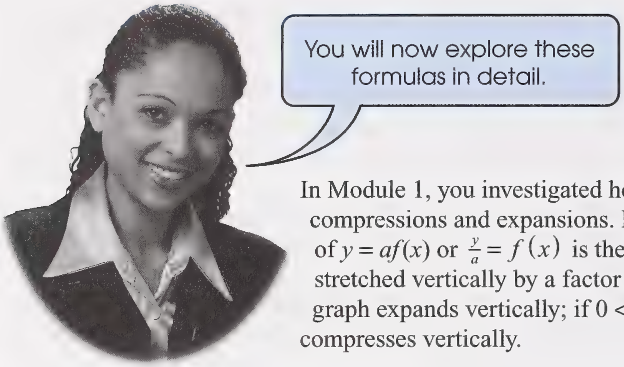
The standard form of an ellipse centred at the origin with a major axis of length $2a$ along the x-axis and a minor axis of length $2b$ along the y-axis is as follows:



Because a and b represent lengths, they are non-negative.

The standard form of an ellipse centred at the origin with a major axis of length $2a$ along the y-axis and a minor axis of length $2b$ along the x-axis is as follows:

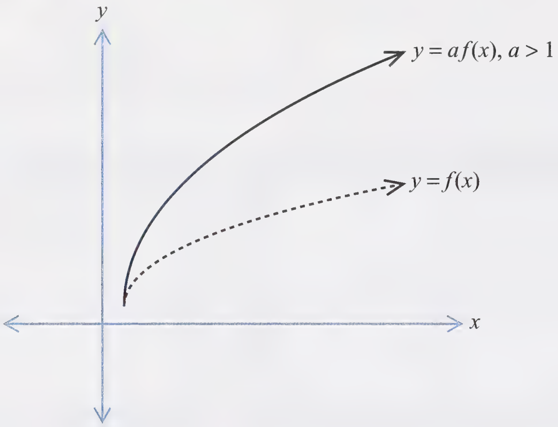



You will now explore these formulas in detail.

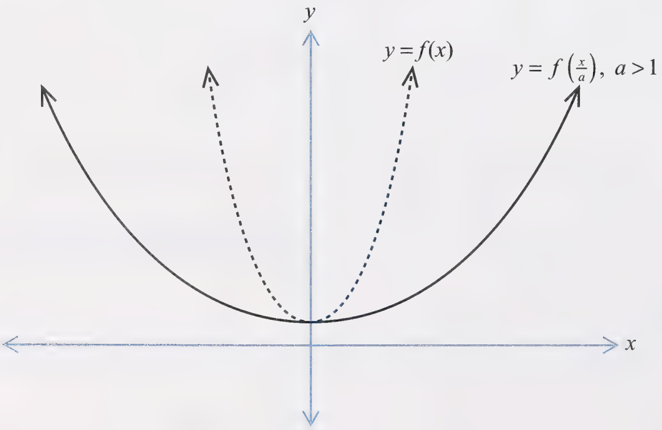
In Module 1, you investigated horizontal and vertical compressions and expansions. Recall that the graph of $y = af(x)$ or $\frac{y}{a} = f(x)$ is the graph of $y = f(x)$ stretched vertically by a factor of a. If $a > 1$, the graph expands vertically; if $0 < a < 1$, the graph compresses vertically.

Mathematical Process

- ■ Communication
- ☒ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ☒ Reasoning
- ■ Technology
- ☒ Visualization

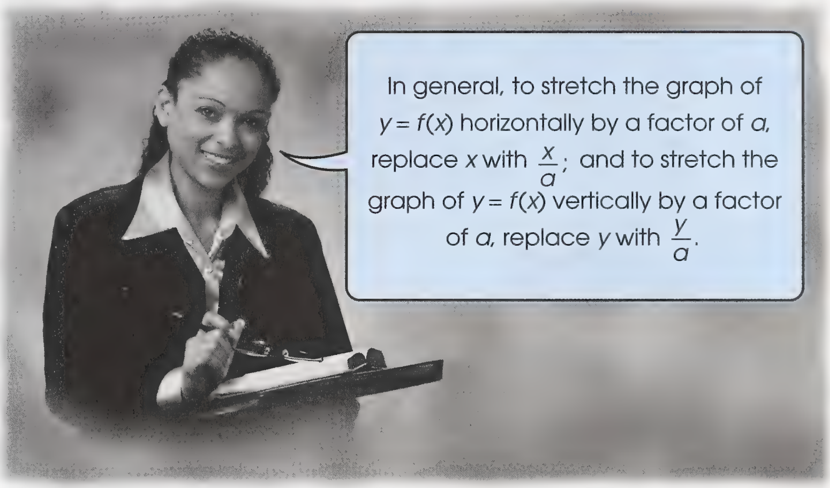


Similarly, the graph of the function $y = f(kx)$ is the graph of $y = f(x)$ stretched horizontally. Recall that if $0 < k < 1$, the graph of $y = f(x)$ is expanded horizontally by a factor of $\frac{1}{k}$. Equivalently, if you replace the parameter k with $\frac{1}{a}$, where $a > 1$, the graph of $y = f\left(\frac{x}{a}\right)$ is the graph of $y = f(x)$ expanded horizontally by a factor of a.



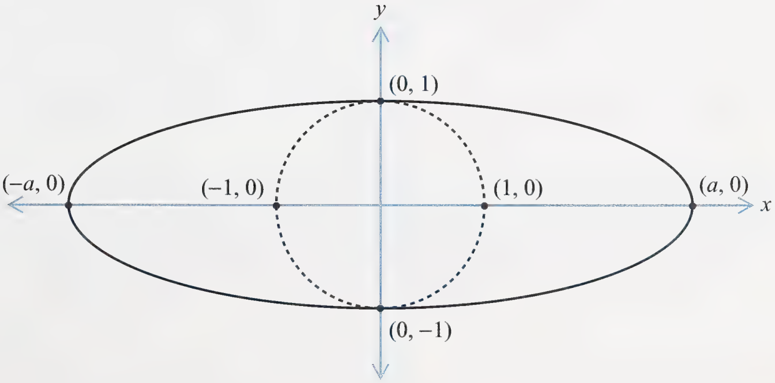
Mathematical Process

☐ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☐ Reasoning
☐ Technology
☒ Visualization


In general, to stretch the graph of $y = f(x)$ horizontally by a factor of a, replace x with $\frac{x}{a}$; and to stretch the graph of $y = f(x)$ vertically by a factor of a, replace y with $\frac{y}{a}$.

Now, think of an ellipse that is formed by stretching a circle horizontally and/or vertically. For example, the circle $x^2 + y^2 = 1$ is centred at the origin with a radius of 1 unit. If you stretch it horizontally by a factor of a, you will obtain an ellipse that crosses the x-axis at $(-a, 0)$ and $(a, 0)$. To determine the equation of this ellipse, substitute $\frac{x}{a}$ for x in $x^2 + y^2 = 1$.

$$x^2 + y^2 = 1$$
$$\left(\frac{x}{a}\right)^2 + y^2 = 1$$
$$\frac{x^2}{a^2} + y^2 = 1$$

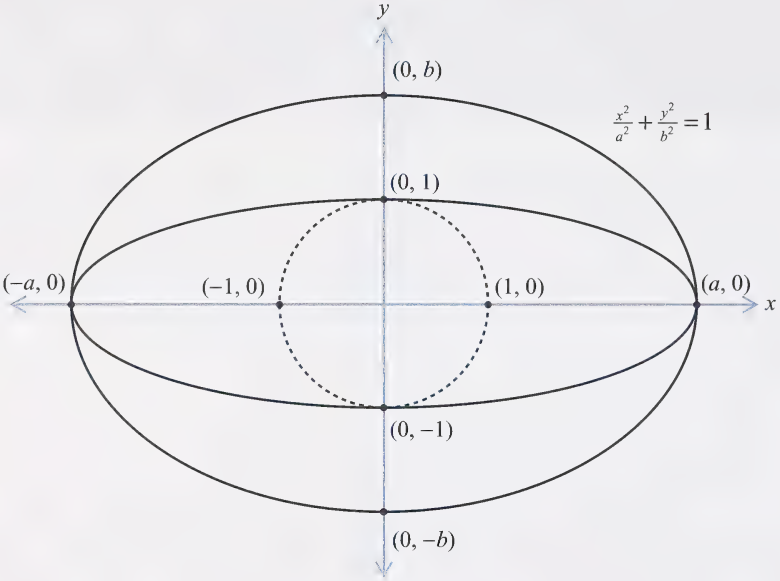


If you stretch this ellipse vertically by a factor of b, you will obtain an ellipse that crosses the x-axis at $(-a, 0)$ and $(a, 0)$ and the y-axis at $(0, b)$ and $(0, -b)$. To determine the equation of this ellipse, substitute $\frac{y}{b}$ for y in $\frac{x^2}{a^2} + y^2 = 1$.

$$\frac{x^2}{a^2} + y^2 = 1$$

$$\frac{x^2}{a^2} + \left(\frac{y}{b}\right)^2 = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



3. Explain how the graph of $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ can be obtained by transforming the graph of $x^2 + y^2 = 1$. Illustrate using a diagram.

Turn to page 114 to compare your response with the one in the Appendix.

Now, practise finding equations of ellipses centred at the origin.

Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☒ Visualization

Example

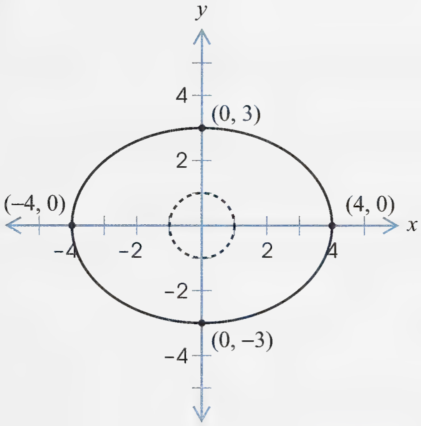
Find the equation of the ellipse with major and minor axes along the coordinate axes, centred at the origin, and crossing the axes at (–4, 0), (4, 0), (0, 3), and (0, –3). Express your answer in both standard form and general form.

Solution

Find the equation in standard form.

Method 1: Transforming the Unit Circle

Sketch a diagram.



The ellipse is the unit circle stretched horizontally by a factor of 4 and vertically by a factor of 3. Therefore, substitute $\frac{x}{4}$ for x and $\frac{y}{3}$ for y in $x^2 + y^2 = 1$.

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \quad \leftarrow \text{standard form}$$

Method 2: Using the Standard Form of an Ellipse

This ellipse is elongated horizontally; therefore, its vertices are on the x-axis. The length of the major axis is the distance between the vertices at (4, 0) and (–4, 0). So, the length of the major axis is 8 units.

$$\begin{aligned} \therefore\ 2a &= 8 \\ a &= 4 \end{aligned}$$

The length of the minor axis is the distance between the points (0, 3) and (0, –3). So, the length of the minor axis is 6 units.

$$\begin{aligned} \therefore\ 2b &= 6 \\ b &= 3 \end{aligned}$$

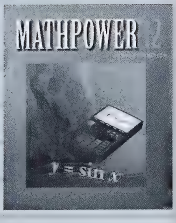
Substitute $a = 4$ and $b = 3$ into the standard form of an ellipse.

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \\ \frac{x^2}{4^2} + \frac{y^2}{3^2} &= 1 \\ \frac{x^2}{16} + \frac{y^2}{9} &= 1 \quad \leftarrow \text{standard form} \end{aligned}$$

To express this equation in general form, rewrite the equation in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

Multiply both sides by the lowest common denominator (LCD), 144.

$$\begin{aligned} 144\left(\frac{x^2}{16} + \frac{y^2}{9}\right) &= 144(1) \\ 9x^2 + 16y^2 &= 144 \\ 9x^2 + 16y^2 - 144 &= 0 \quad \leftarrow \text{general form} \end{aligned}$$



4. Turn to page 151 of *MATHPOWER 12* and answer question 32 of "Applications and Problem Solving."

5. Use transformations of the unit circle to find the equation of the following ellipses in standard form and general form. Each ellipse is centred at the origin and has major and minor axes along the coordinate axes.

   a. The ellipse passes through $(-1, 0)$, $(1, 0)$, $(0, 2)$, and $(0, -2)$.
   b. The ellipse passes through $(-8, 0)$, $(8, 0)$, $(0, 1)$, and $(0, -1)$.

Turn to page 114 to compare your responses with those in the Appendix.

Every ellipse you have dealt with so far has been centred at the origin. What if an ellipse is translated horizontally *h* units and vertically *k* units so its centre is at (*h, k*)?

Recall from Module 1 that if the function $y = f(x)$ undergoes a horizontal and vertical translation, its equation becomes $y - k = f(x - h)$ or $y = f(x - h) + k$.

Similarly, when the centre of an ellipse is moved from the origin to point (h, k), the standard forms of the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, become

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \text{ and } \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

Example

Mathematical Process

☐ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☒ Problem Solving
☒ Reasoning
☐ Technology
☒ Visualization

a. Describe the transformations necessary to transform the circle $x^2 + y^2 = 1$ into an ellipse centred at (3, 4), with a major axis 6 units in length and parallel to the x-axis, and a minor axis 4 units in length and parallel to the y-axis. State the equation of the ellipse in standard form and general form.

b. Draw the graphs of the unit circle and the ellipse.

c. What is the domain and range of the ellipse?

Solution

a. The radius of the unit circle is 1 unit. Therefore, its diameter is 2 units.

To transform this circle into an ellipse with a major axis 6 units long and parallel to the x-axis, the circle must be stretched horizontally by a factor of 3.

To transform this circle into an ellipse with a minor axis 4 units long and parallel to the y-axis, the circle must be stretched vertically by a factor of 2.

To move the centre from the origin to (3, 4), the circle must be translated 3 units to the right and 4 units up.

To obtain the equation of the ellipse, transform the equation of the unit circle, $x^2 + y^2 = 1$.

First, find the equation of the ellipse in standard form by transforming the equation of the unit circle.

To stretch the circle horizontally by a factor of 3 and vertically by a factor of 2, substitute $\frac{x}{3}$ for x and $\frac{y}{2}$ for y into the equation of the unit circle.

$$\begin{aligned} x^2 + y^2 &= 1 \\ \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 &= 1 \\ \frac{x^2}{9} + \frac{y^2}{4} &= 1 \end{aligned}$$

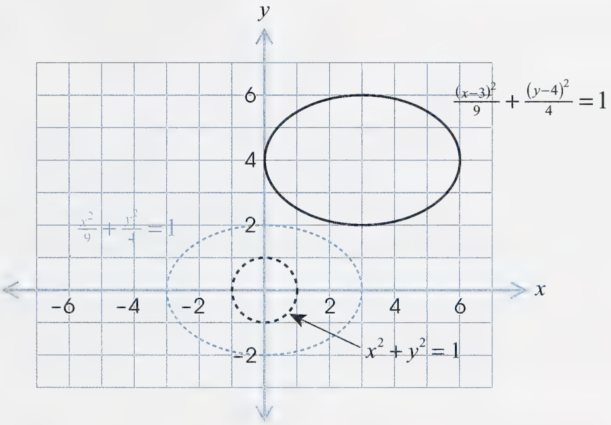
To translate this ellipse 3 units to the right and 4 units up, substitute $(x-3)$ for x and $(y-4)$ for y.

$$\frac{(x-3)^2}{9}+\frac{(y-4)^2}{4}=1 \quad \leftarrow \text{standard form}$$

Now, express the equation in general form.

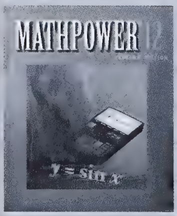
$$\begin{aligned}
\frac{(x-3)^2}{9}+\frac{(y-4)^2}{4}&=1\\
36\left[\frac{(x-3)^2}{9}+\frac{(y-4)^2}{4}\right]&=36(1) \quad \leftarrow \text{Multiply both sides by the LCD, 36.}\\
4(x-3)^2+9(y-4)^2&=36\\
4(x^2-6x+9)+9(y^2-8y+16)&=36 \quad \leftarrow \text{Expand.}\\
4x^2-24x+36+9y^2-72y+144&=36\\
4x^2+9y^2-24x-72y+144&=0 \quad \leftarrow \text{general form}
\end{aligned}$$

b.



c. The horizontal axis extends from (0, 4) to (6, 4). Therefore, the domain of the ellipse is $0 \le x \le 6$.

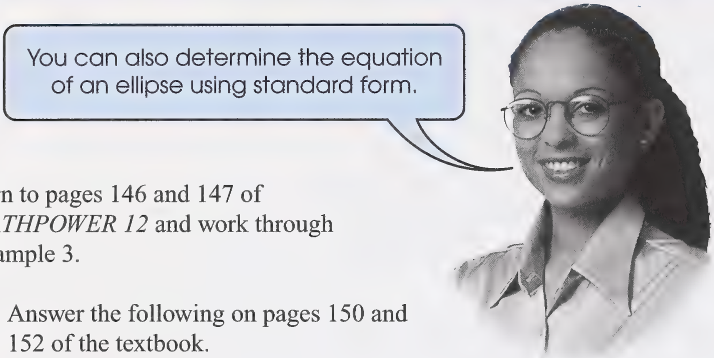
The vertical axis extends from (3, 2) to (3, 6). Therefore, the range of the ellipse is $2 \le y \le 6$.

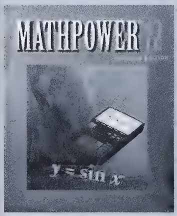


Turn to page 144 of *MATHPOWER 12* and read from the red line following Example 1 to the red line on page 145.

6. Answer questions 33, 34, and 36 of “Applications and Problem Solving” on page 151 of the textbook.

Turn to page 118 to compare your responses with those in the Appendix.


You can also determine the equation of an ellipse using standard form.



Turn to pages 146 and 147 of *MATHPOWER 12* and work through Example 3.

7. Answer the following on pages 150 and 152 of the textbook.

- **a.** question 21 of “Practice”
- **b.** question 40.a. of “Applications and Problem Solving”

Turn to page 120 to compare your responses with those in the Appendix.

Have you noticed that when the equation of an ellipse with horizontal and vertical axes is written in general form, $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where $A \neq C$ and A and C are non-zeros with the same sign?

For example, $\frac{(x-3)^2}{9} + \frac{(y-4)^2}{4} = 1$ in general form is $4x^2 + 9y^2 - 24x - 72y + 144 = 0$ or $-4x^2 - 9y^2 + 24x + 72y - 144 = 0$. Notice that $A \neq C$ in both equations ($A = 4$ and $C = 9$ or $A = -4$ and $C = -9$) and that $AC > 0$.

In general, given an equation of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where $A \neq C$ and $AC > 0$, the equation represents an ellipse or one of its degenerate forms.

At this point if you are given an equation in general form, you should know whether it is a circle or an ellipse, be able to express the equation in standard form, and be able to graph the equation if it exists.

Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☒ Visualization

Example

Given $4x^2 + 25y^2 - 8x + 150y + 129 = 0$, do the following.

a. Express the equation in standard form.
b. Draw the ellipse.

Solution

a.

$$4x^2+25y^2-8x+150y+129=0$$

$$4x^2-8x+25y^2+150y=-129 \quad \leftarrow \text{Group the terms.}$$

$$4\left(x^2-2x\right)+25\left(y^2+6y\right)=-129 \quad \leftarrow \text{Remove the coefficients from } x^2 \text{ and } y^2.$$

$$4\left[\left(x^2-2x+1\right)-1\right]+25\left[\left(y^2+6y+9\right)-9\right]=-129 \quad \leftarrow \text{Complete the squares.}$$

$$4\left[(x-1)^2-1\right]+25\left[(y+3)^2-9\right]=-129$$

$$4(x-1)^2-4+25(y+3)^2-225=-129$$

$$4(x-1)^2+25(y+3)^2=100$$

$$\frac{\overset{1}{\cancel{4}}(x-1)^2}{\underset{25}{\cancel{100}}}+\frac{\overset{1}{\cancel{25}}(y+3)^2}{\underset{4}{\cancel{100}}}=\frac{100}{100} \quad \leftarrow \text{Divide each term by 100.}$$

$$\frac{(x-1)^2}{25}+\frac{(y+3)^2}{4}=1 \quad \leftarrow \text{Simplify.}$$

Note: Standard form is also called completed square form.

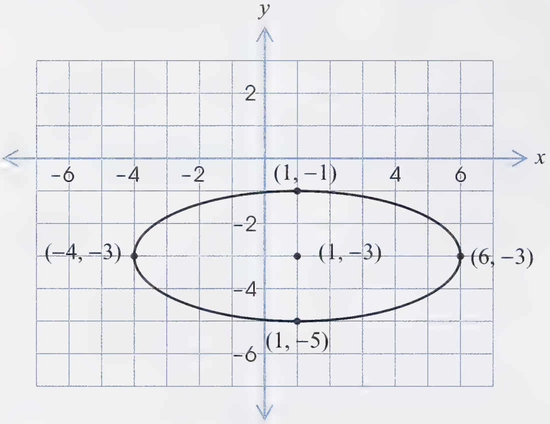
b. Compare $\frac{(x-1)^2}{25}+\frac{(y+3)^2}{4}=1$ to $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

$\therefore\ h=1,\ k=-3,\ a=\sqrt{25}=5,$ and $b=\sqrt{4}=2$

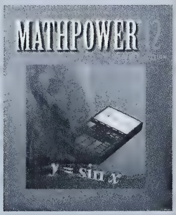
The centre of this ellipse is at (*h*, *k*) or (1, −3).

The major axis is parallel to the *x*-axis, and its endpoints are $a = 5$ units from the centre.

The minor axis is parallel to the *y*-axis, and its endpoints are $b = 2$ units from the centre.



Now, practise completing the square to convert the equations from general form to standard form.

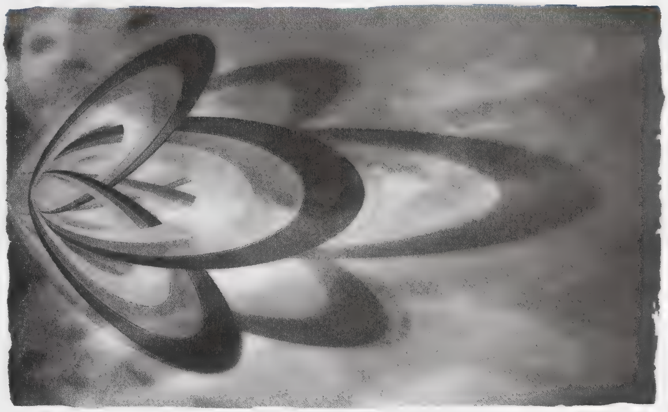


Turn to pages 147 and 148 of *MATHPOWER 12* and work through Example 5.
Note: In Pure Mathematics 30 you are not responsible for locating foci.

8. Express each equation in standard form, and draw the graph if it exists.

a. $x^2 + 4y^2 + 2x - 8y + 1 = 0$
b. $4x^2 + 9y^2 - 8x - 54y + 49 = 0$
c. $4x^2 + y^2 + 4y = 0$

Turn to page 122 to compare your responses with those in the Appendix.



Ellipses, like other relations, can be stretched about lines other than the coordinate axes. From the Pure Mathematics 30 Multimedia CD, view the animation *Stretching an Ellipse About $y = -4$*. This animation shows the results of stretching $\frac{(x-1)^2}{5^2} + \frac{(y+3)^2}{2^2} = 1$ vertically by a factor of 3 about the line $y = -4$. Notice how the distance from the line to the new ellipse is three times the distance to the given ellipse.

This stretch can be done algebraically as well. Just remember to translate the equation so the stretch line matches an axis, and stretch the new equation. Then translate the result so the stretch line is back in its original place.

9. Determine the equation of the stretched ellipse in the animation.

Turn to page 125 to compare your response with the one in the Appendix.

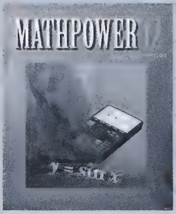
Now Try This

You have just rewritten equations from general form to standard form. At each step in the process, you had to be careful when writing equivalent forms of these equations or else you would end up with a different ellipse than the one represented by the original equation. In the next problem you will have to use your abilities in writing equations.

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ☒ Problem Solving
- ☒ Reasoning
- ■ Technology
- ■ Visualization

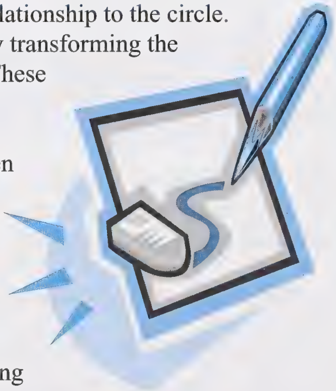
10. Turn to page 152 of *MATHPOWER 12* and answer "NUMBER POWER."



Turn to page 126 to compare your response with the one in the Appendix.

Looking Back

In this activity you explored the ellipse and its relationship to the circle. You examined how the ellipse can be obtained by transforming the equation of the unit circle centred at the origin. These transformations led to the standard form of the ellipse. From its standard form, you were able to identify the critical features of the ellipse and then use those features to sketch its graph. You also examined the general form of the equation of the ellipse and discovered how to convert from general form into standard form by completing the square.



Mathematical Process

- ☒ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

In your journal, compare the process of completing the square for the circle with the process of completing the square for the ellipse. How are they similar? How do they differ? Illustrate your answers with examples.

Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

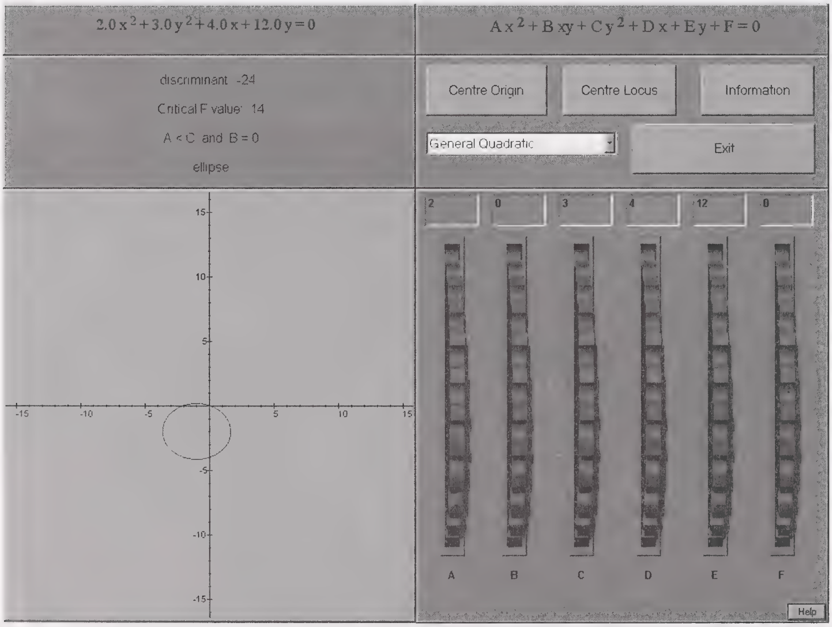
Among the key concepts of Section 1 are the ability to convert equations in general form to standard form, and vice versa, and the ability to use the standard form to graph and analyze circles and ellipses.

From the Pure Mathematics 30 Companion CD, open *The Quadratic Relations Explorer* in the Explorers folder.

Note: The graphics and procedures that follow are for the Windows version. However, the graphics and procedures for the Macintosh version are similar.

When you open the explorer, the initial screen allows you to alter the coefficients of the general form, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. You can either change the values of the coefficients by clicking and dragging the wheels or by simply clicking in the rectangles above the wheels and entering the desired values. Once you enter a value, the graph will automatically change.

For example, if you enter the equation $2x^2 + 3y^2 + 4x + 12y = 0$, the following display appears.



$2.0x^2 + 3.0y^2 + 4.0x + 12.0y = 0$

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

discriminant -24

Critical F value: 14

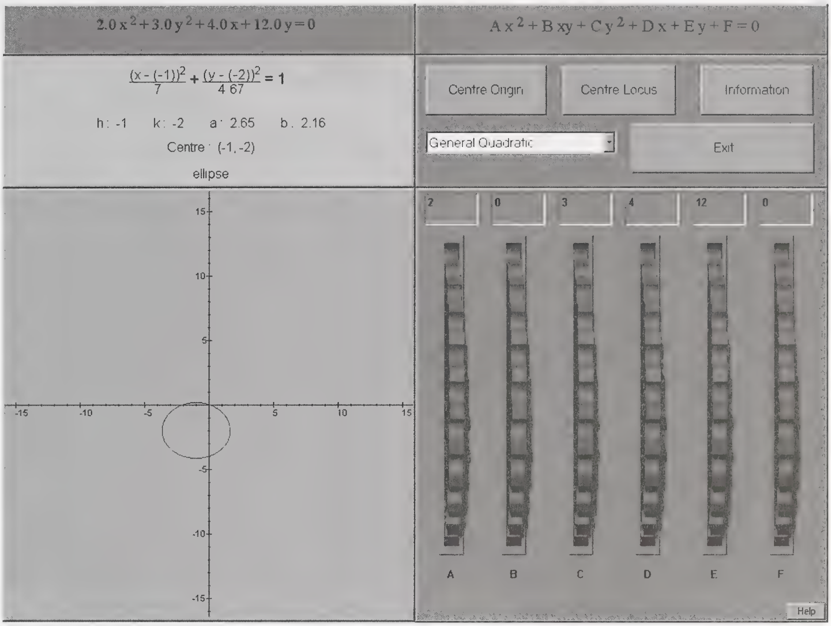
A < C and B = 0

ellipse

General Quadratic

A	B	C	D	E	F
2	0	3	4	12	0

To convert the general form of the ellipse to standard form, simply right-click the rectangle above the graph with your mouse. The following display should appear.



$2.0x^2+3.0y^2+4.0x+12.0y=0$

$$\frac{(x-(-1))^2}{7}+\frac{(y-(-2))^2}{4.67}=1$$

h: -1 k: -2 a: 2.65 b: 2.16

Centre: (-1, -2)

ellipse

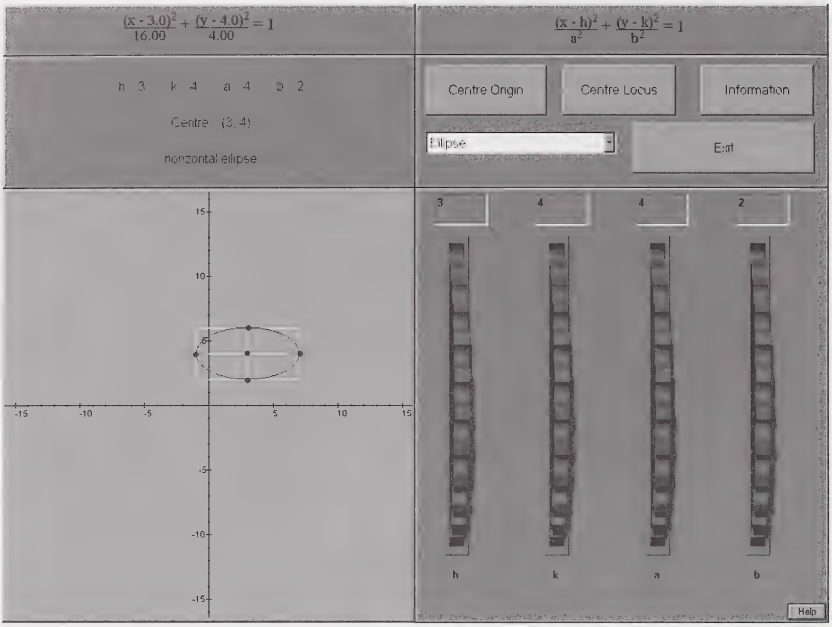
General Quadratic

A: 2, B: 0, C: 3, D: 4, E: 12, F: 0

Note: To continue, simply right-click the mouse on the rectangle again.

You can also enter the equation of a circle or ellipse given in standard form. Use the pull-down menu on the rectangle titled "General Quadratic" to select "Circle" or "Ellipse." If you select "Ellipse," for example, you can enter the values of *h*, *k*, *a*, and *b* in the same way as the coefficients for the general form.

This display shows an example of an equation of an ellipse in standard form.



$$\frac{(x-3.0)^2}{16.00}+\frac{(y-4.0)^2}{4.00}=1$$

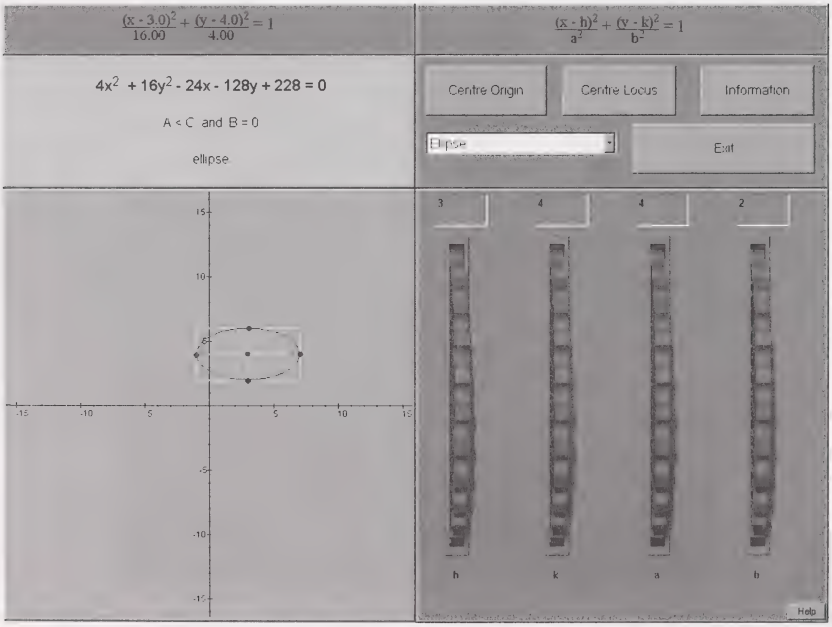
$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

h 3 k 4 a 4 b 2

Centre (3, 4)

horizontal ellipse

You can also change the form of this ellipse from standard form to general form by right-clicking the rectangle above the graph. The following display should appear.



$$\frac{(x-3.0)^2}{16.00}+\frac{(y-4.0)^2}{4.00}=1$$

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$4x^2 + 16y^2 - 24x - 128y + 228 = 0$

$A < C$ and $B = 0$

ellipse

Notice that when you use *The Quadratic Relations Explorer* to convert an equation from standard form to general form, the result often can be reduced by dividing by a common factor. In the preceding case, the answer given was $4x^2 + 16y^2 - 24x - 128y + 228 = 0$. Because each coefficient is a multiple of 4, you can divide each term by 4 to reduce the equation.

$$\frac{4x^2}{4} + \frac{16y^2}{4} - \frac{24x}{4} - \frac{128y}{4} + \frac{228}{4} = \frac{0}{4}$$
$$x^2 + 4y^2 - 6x - 32y + 57 = 0$$

Remember: There are an infinite number of ways of writing the equation of a particular ellipse. Simply, multiply or divide each term in the equation by a non-zero constant.

Use *The Quadratic Relations Explorer* to check your answers to the following questions.

1. What are the domain and range of the ellipse $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$? Express the equation in general form.
2. Write $x^2 + y^2 + 2x - 4y - 20 = 0$ in standard form. What are the domain and range of this circle?
3. What is the equation of the ellipse with centre (2, 3), a horizontal axis 4 units in length, and a vertical axis 6 units in length? Write the equation in both standard form and general form.

Turn to page 126 to compare your responses with those in the Appendix.

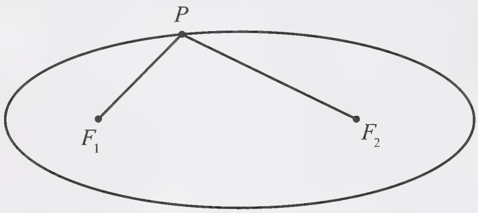
Enrichment

In Section 1 you derived the standard form of an ellipse by transforming the standard form of the unit circle. You can also obtain the same result by using the Locus Definition of the ellipse. *Locus* is the Latin word that is the root of the English word *location*. The locus definition tells you how the points on an ellipse are located relative to two fixed points called foci.

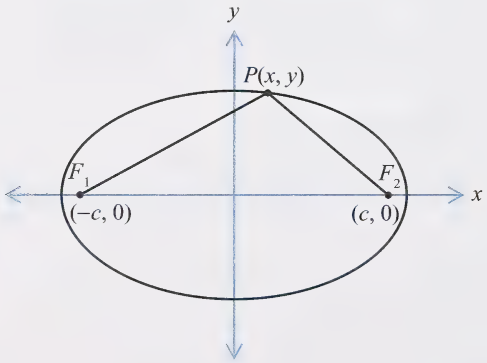
Locus Definition

An ellipse is a set of points such that the sum of the distances each point lies from two fixed points is constant.

In the diagram, F_1 and F_2 are the foci of the ellipse and $PF_1 + PF_2$ is constant, regardless of the position of point P on the curve.



Suppose F_1 and F_2 are located at $(-c, 0)$ and $(c, 0)$ on the x-axis (as shown) and $PF_1 + PF_2 = 2a$. What is the equation of this ellipse?



Because $PF_1 + PF_2 = 2a$.

$$\sqrt{\left[x-(-c)\right]^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$$

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$

$$\left(\sqrt{(x+c)^2+y^2}\right)^2=\left(2a-\sqrt{(x-c)^2+y^2}\right)^2$$

$$(x+c)^2+\cancel{y}^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+\cancel{y}^2$$

$$(x+c)^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2$$

$$\cancel{x}^2+2cx+\cancel{c}^2=4a^2-4a\sqrt{(x-c)^2+y^2}+\cancel{x}^2-2cx+\cancel{c}^2$$

$$\cancel{4}cx=\cancel{4}a^2-\cancel{4}a\sqrt{(x-c)^2+y^2}$$

$$cx=a^2-a\sqrt{(x-c)^2+y^2}$$

$$a\sqrt{(x-c)^2+y^2}=a^2-cx$$

$$\left(a\sqrt{(x-c)^2+y^2}\right)^2=\left(a^2-cx\right)^2$$

$$a^2\left[(x-c)^2+y^2\right]=\left(a^2-cx\right)^2$$

$$a^2\left(x^2-2cx+c^2+y^2\right)=a^4-2a^2cx+c^2x^2$$

$$a^2x^2-\cancel{2a^2cx}+a^2c^2+a^2y^2=a^4-\cancel{2a^2cx}+c^2x^2$$

$$a^2x^2+a^2c^2+a^2y^2=a^4+c^2x^2$$

$$a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2$$

$$\left(a^2-c^2\right)x^2+a^2y^2=a^2\left(a^2-c^2\right)$$

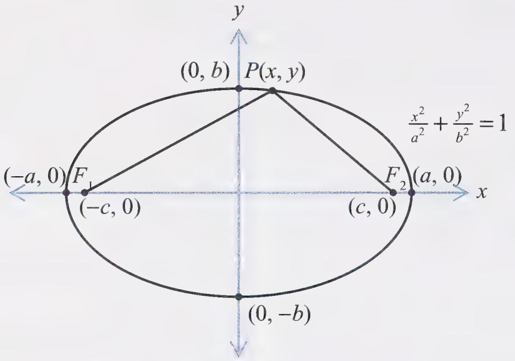
Now, substitute b^2 for a^2-c^2.

$$b^2x^2+a^2y^2=a^2b^2$$

$$\frac{\cancel{b}^2x^2}{a^2\cancel{b}^2}+\frac{\cancel{a}^2y^2}{\cancel{a}^2b^2}=\frac{a^2b^2}{a^2b^2}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

You should recognize this equation as the standard form of an ellipse centred at the origin and crossing the x-axis at $(\pm a, 0)$ and the y-axis at $(0, \pm b)$.



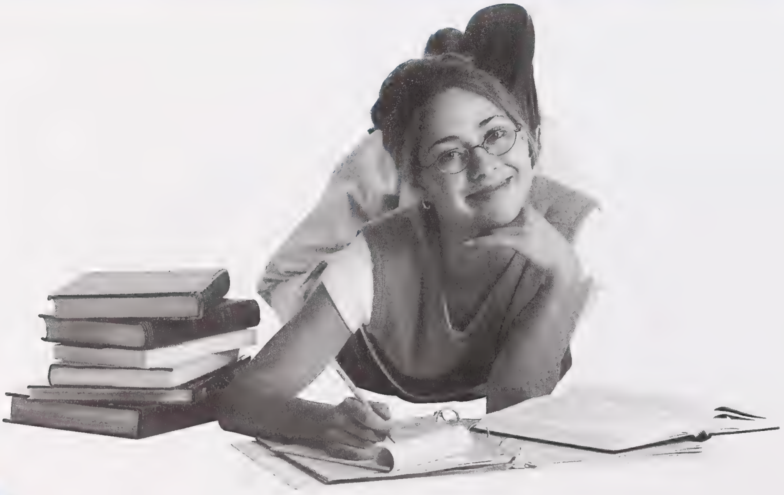
The relationship $b^2 = a^2 - c^2$ or $a^2 = b^2 + c^2$ provides a relationship among the distance, c, the foci lie from the centre, the length, a, of the semi-major axis, and the length, b, of the semi-minor axis.

Now, try your hand at the following.

1. If the sum of the distances each point on an ellipse lies from the foci is 10, find the standard form of the equation of the ellipse with foci at (0, 3) and (0, −3).

2. Determine the foci of $\frac{x^2}{169} + \frac{y^2}{25} = 1$.

Turn to page 130 to compare your responses with those in the Appendix.



Conclusion

In this section you obtained the conic sections—the circle, the ellipse, the parabola, and the hyperbola and their degenerate forms—by slicing a double-napped, right-circular cone or by slicing a cylinder.

You then investigated the circle and the ellipse, their graphs, and the standard form and general form of their equations. You converted the equations from standard form into general form, and vice versa, in order to analyze the graphs of these curves.

Finally, you discovered that the ellipse can be obtained by transforming the unit circle—stretching and translating it horizontally and vertically.

Did you know that the orbits of the planets about the Sun are elliptical? This fact was discovered by Johannes Kepler. He analyzed the meticulous observations of another astronomer by the name of Tycho Brahe. Kepler first published his findings in 1609.

For more information on Johannes Kepler and planetary motion, visit the following website:

http://www-history.mcs.st-and.ac.uk/Biographies/Kepler.html

Assignment

Turn to Assignment Booklet 4A and complete the assignment for Section 1. Remember to submit your completed Assignment Booklet for assessment before starting Section 2.

Section 2

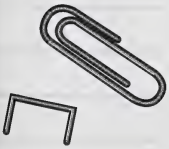
ROBERT HOLMES/CORBIS

Hyperbolas and Parabolas

Have you been to Calgary and attended a hockey game, concert, or any other event in the Pengrowth Saddledome? This award-winning design was completed in 1983 at a cost of $176 million. The Saddledome was home to some of the events in the 1988 Winter Olympics. In fact, it was the first arena in North America with the capability to fit the larger, international-sized ice surface.

The roof of the Saddledome is a perfect example of a reverse hyperbolic paraboloid. A hyperbolic paraboloid is a geometric surface with three axes of symmetry that intersect at what mathematicians call its saddle point, because the curve at this point resembles a saddle. The hyperbolic paraboloid is so named because horizontal planes that intersect the surface produce hyperbolas and vertical planes that intersect the surface produce parabolas.

In this section you will explore hyperbolas and parabolas. You will investigate their equations, graphs, and the transformations necessary to produce them. You will then discover other examples in architecture and design where these curves occur.



Activity 1: The Hyperbola

Mathematical Process

☐ Communication
☒ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☐ Reasoning
☐ Technology
☐ Visualization

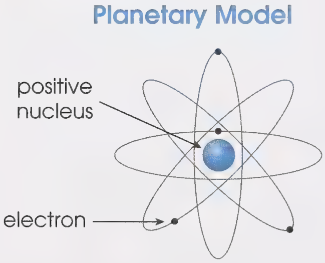
Ernest Rutherford, physicist and Nobel laureate for his work on radioactivity, proposed the planetary model of the atom in 1911. Prior to Rutherford's proposal, most physicists thought of the atom as a plum pudding, where the electrons (the plums) were dispersed throughout a blob of positive and relatively light pudding.

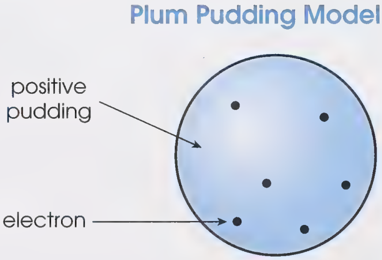
BETTMAN/CORBIS

Ernest Rutherford (1871–1937)

Rutherford and his colleagues fired positively charged alpha particles from a radioactive source towards gold film. He knew that if the plum-pudding model was correct, the alpha particles would pass straight through the pudding. However, Rutherford observed that, while many passed straight through the atom, a significant number of alpha particles were deflected through a variety of angles—some actually rebounded.

Rutherford theorized that the particles were deflected by a dense concentration of positive charges at the nucleus and that electrons orbited the nucleus, just like a miniature planetary system. This was an important step towards the quantum mechanics model.



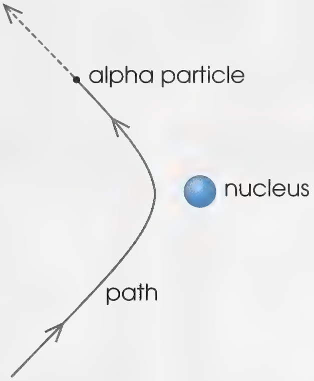


You can find more information about Ernest Rutherford and his experiments at the following website:

http://www.nobelprize.org/nobel_prizes/chemistry/laureates/1908/rutherford-bio.html

hyperbola: the conic produced by slicing a right-circular cone parallel to its axis or at an angle less than the angle of the generator

The path along which the deflected alpha particle travels in Rutherford's experiment can be modelled by one branch of a **hyperbola**.



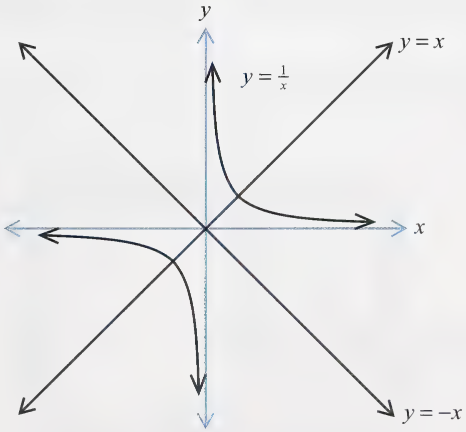
In this activity you will explore hyperbolas. Specifically, you will investigate their equations and graphs. In your analysis you will look at equations in both general form and standard form, convert between forms, and use these forms to determine how one hyperbola can be transformed into another.

Mathematical Process

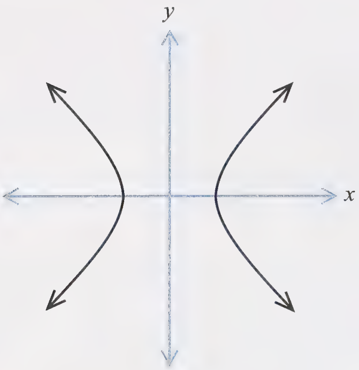
- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☒ Visualization

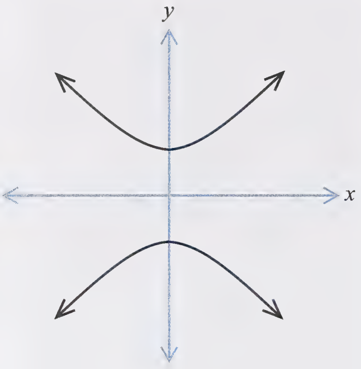
In Pure Mathematics 20 and in Module 1 of this course, you graphed, transformed, and analyzed the reciprocal function, $y=\frac{1}{x}$. You discovered that the graph of a reciprocal function was a hyperbola, with asymptotes along the coordinate axes. Recall that asymptotes are boundary lines a graph approaches as you move outward from the origin in the coordinate plane.

The graph of $y=\frac{1}{x}$ has two axes of symmetry: the line $y=x$ and the line $y=-x$.



In this activity you will examine hyperbolas with vertical and horizontal axes of symmetry only.





Example

Graph $x^2 - y^2 = 1$ on your graphing calculator. What are the asymptotes of the graph?

Mathematical Process

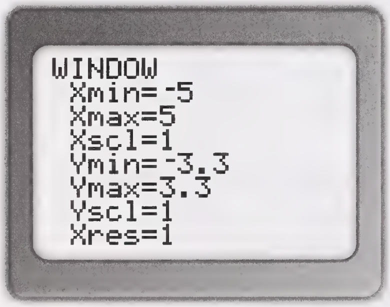
- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☒ Technology
- ☐ Visualization

Solution

Express $x^2 - y^2 = 1$ in the form $y = f(x)$.

$$\begin{aligned} x^2 - y^2 &= 1 \\ x^2 - 1 &= y^2 \\ y^2 &= x^2 - 1 \\ y &= \pm\sqrt{x^2 - 1} \end{aligned}$$

To display detail near the x-intercepts and minimize distortion, use the following window settings.

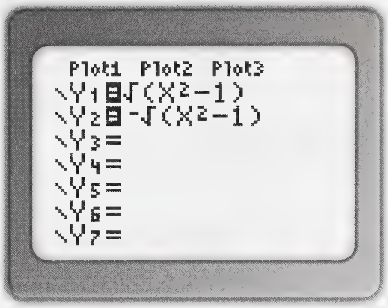


```
WINDOW
 Xmin=-5
 Xmax=5
 Xscl=1
 Ymin=-3.3
 Ymax=3.3
 Yscl=1
 Xres=1
```

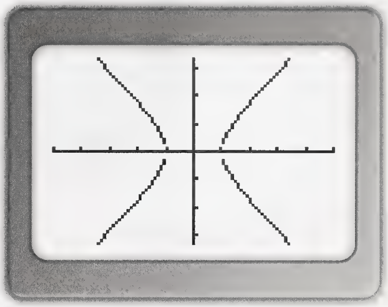
Next, enter $y=\sqrt{x^2-1}$ as Y_1 and $y=-\sqrt{x^2-1}$ as Y_2.

Y= 2nd [√] X,T,Θ,n x^2 − 1) ENTER (−)

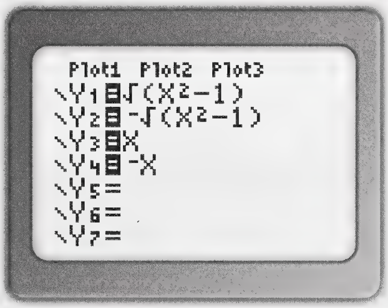
2nd [√] X,T,Θ,n x^2 − 1) GRAPH



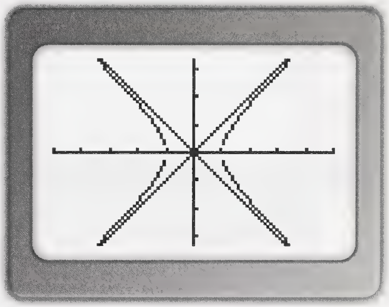
```
Plot1 Plot2 Plot3
\Y1=√(X²-1)
\Y2=-√(X²-1)
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



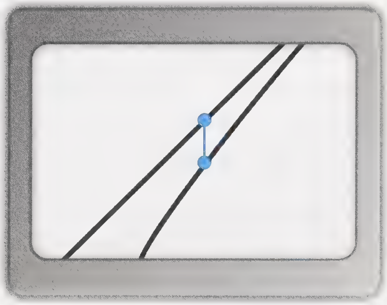
The asymptotes appear to be $y=x$ and $y=-x$. Graph these lines as Y_3 and Y_4.



```
Plot1 Plot2 Plot3
\Y1=√(X²-1)
\Y2=-√(X²-1)
\Y3=X
\Y4=-X
\Y5=
\Y6=
\Y7=
```



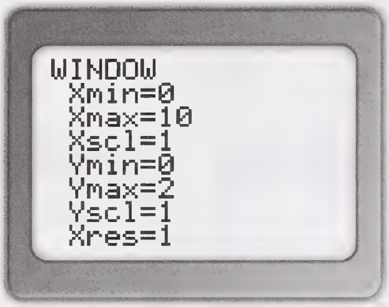
As you go outward from the origin, the branches of the hyperbola appear to approach these lines. Consider $y=x$ and the portion of the hyperbola in the first quadrant obtained from $y=\sqrt{x^2-1}$.



The vertical distance between the two graphs is

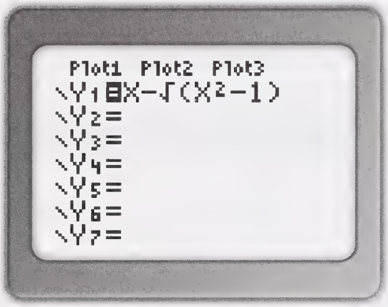
$$y_1-y_2=x-\sqrt{x^2-1}$$

Graph this difference on your graphing calculator using the following window settings.

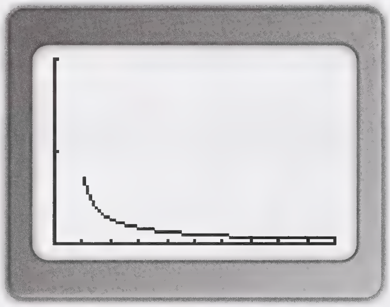


```
WINDOW
 Xmin=0
 Xmax=10
 Xscl=1
 Ymin=0
 Ymax=2
 Yscl=1
 Xres=1
```

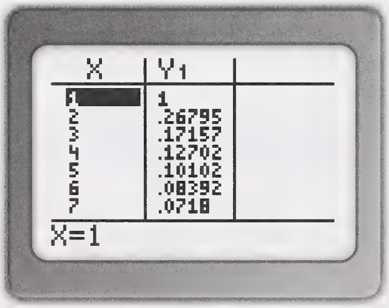
Y= X,T,Θ,*n* − 2nd [√] X,T,Θ,*n* x^2 − 1) GRAPH



```
Plot1 Plot2 Plot3
\Y1■X-√(X²-1)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



Check the table of values to verify that this difference approaches 0 as x increases.



X	Y1
1	1
2	.26795
3	.17157
4	.12702
5	.10102
6	.08392
7	.0718

X=1

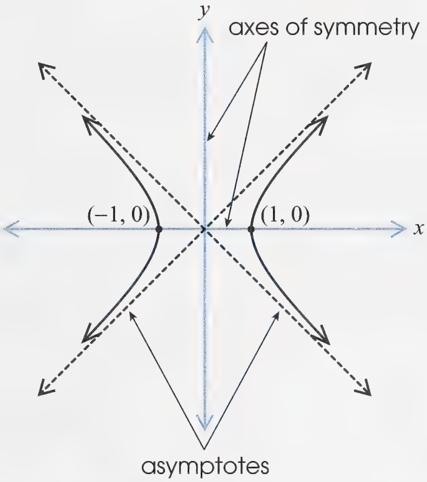
Using this method, you can verify that both branches of the hyperbola lie between and approach lines $y = x$ and $y = -x$.

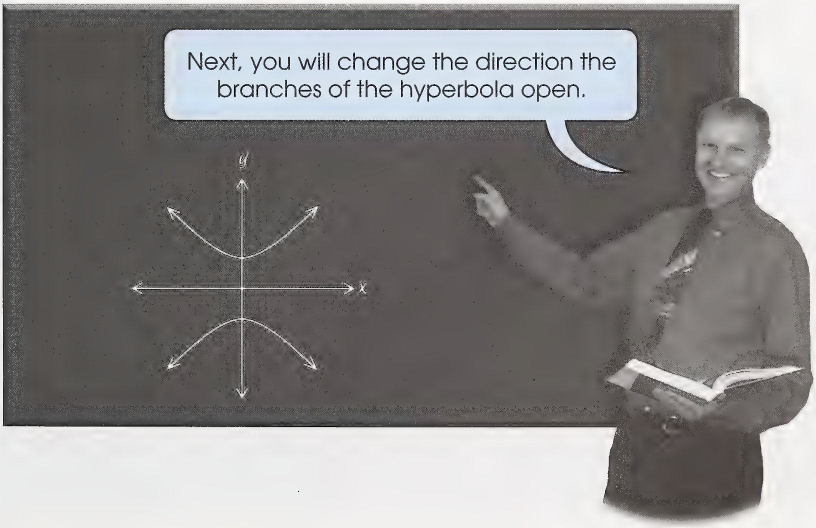
transverse axis: the axis of symmetry joining the vertices of a hyperbola

conjugate axis: the axis of symmetry perpendicular to the transverse axis of a hyperbola

The hyperbola $x^2 - y^2 = 1$ crosses the x-axis at $(-1, 0)$ and $(1, 0)$. These points on the horizontal axis of symmetry are the **vertices** of this hyperbola. The axis of symmetry that joins the vertices is called the **transverse axis**. The vertical axis of symmetry, which lies along the y-axis in this case, is perpendicular to the transverse axis and is called the **conjugate axis**.

The two axes of the hyperbola intersect at the **centre** of the hyperbola. In this case the centre is at the origin.




Next, you will change the direction the branches of the hyperbola open.
y
x

Mathematical Process

■ Communication
■ Connection
■ Estimation
■ Mental Math
■ Problem Solving
☒ Reasoning
■ Technology
☒ Visualization

Example

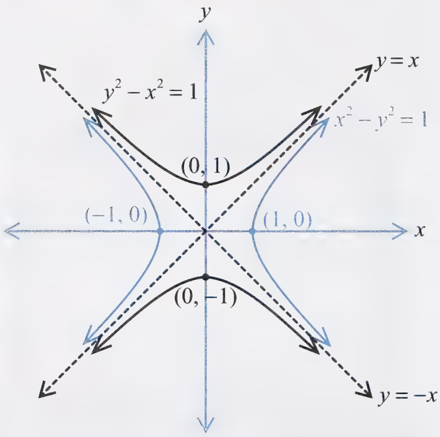
Describe and graph the inverse of $x^2 - y^2 = 1$.

Solution

Interchange the variables to form the inverse of $x^2 - y^2 = 1$. Thus, the inverse is $y^2 - x^2 = 1$.

Recall that the graph of an inverse is the reflection of the original curve in the line $y = x$. Because the line $y = x$ is an asymptote of the original hyperbola, the inverse is simply the graph of the original hyperbola rotated 90°.

The asymptotes of $x^2 - y^2 = 1$ are $y = x$ and $y = -x$. The vertices are (0, 1) and (0, −1); the transverse axis lies along the y-axis; the conjugate axis lies along the x-axis; and the centre of the hyperbola is (0, 0).



1. Determine the domain and range of each hyperbola.

   a. $x^2 - y^2 = 1$
   b. $y^2 - x^2 = 1$

Turn to page 131 to compare your responses with those in the Appendix.

Now, you will transform the graphs of $x^2 - y^2 = 1$ and $y^2 - x^2 = 1$ to obtain other hyperbolas.

Mathematical Process

■ Communication
■ Connection
■ Estimation
■ Mental Math
■ Problem Solving
☒ Reasoning
■ Technology
☒ Visualization

Example

Describe the transformations necessary to obtain the graph of the following hyperbola from the graph of $x^2 - y^2 = 1$. Describe and sketch the curve.

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Solution

Because $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is the same as $\left(\frac{x}{4}\right)^2 - \left(\frac{y}{3}\right)^2 = 1$, the graph of $x^2 - y^2 = 1$ is stretched horizontally by a factor of 4 and vertically by a factor of 3.

Because the hyperbola opens to the left and to the right, use the horizontal stretch factor of 4 to locate the vertices.

$$(-1 \times 4, 0) = (-4, 0) \text{ and } (1 \times 4, 0) = (4, 0)$$

The slopes of the asymptotes of $x^2 - y^2 = 1$, using $\frac{\text{rise}}{\text{run}}$ are $\pm\frac{1}{1}$. Therefore, the slopes of the new asymptotes are

$$\begin{aligned} m &= \pm\frac{1 \times \text{vertical stretch}}{1 \times \text{horizontal stretch}} \\ &= \pm\frac{3}{4} \end{aligned}$$

Because the asymptotes pass through the origin, their equations are of the form $y = mx$. Thus, the asymptotes are $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$.

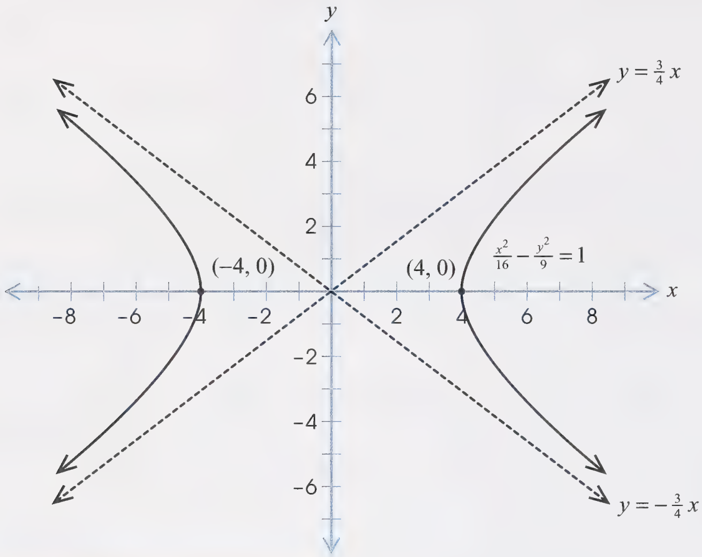
The asymptotes could also be obtained from the original asymptotes, $y = \pm x$, by applying the same transformations the hyperbola underwent.

Substitute $\frac{x}{4}$ for x and $\frac{y}{3}$ for y in $y = \pm x$.

$$\begin{aligned} y &= \pm x \\ \frac{y}{3} &= \pm\frac{x}{4} \\ y &= \pm\frac{3}{4}x \end{aligned}$$

Thus, the new asymptotes are $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$.

Sketch the graph by first locating the vertices. Then draw the asymptotes using their slopes. Finally, sketch the branches of the hyperbola opening to the left and to the right from the vertices and between the asymptotes.



Example

Describe the transformations necessary to obtain the graph of $\frac{y^2}{4} - \frac{x^2}{9} = 1$ from the graph of $y^2 - x^2 = 1$. Describe and sketch the curve.

Mathematical Process
- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☒ Visualization

Solution

Because $\frac{y^2}{4} - \frac{x^2}{9} = 1$ is the same as $\left(\frac{y}{2}\right)^2 - \left(\frac{x}{3}\right)^2 = 1$, the graph of $y^2 - x^2 = 1$ is stretched horizontally by a factor of 3 and vertically by a factor of 2.

Because the hyperbola opens upward and downward, use the vertical stretch factor of 2 to locate the vertices.

$$(0, -1 \times 2) = (0, -2) \text{ and } (0, 1 \times 2) = (0, 2)$$

The slopes of the asymptotes of $y^2 - x^2 = 1$, using $\frac{\text{rise}}{\text{run}}$, are $\pm\frac{1}{1}$. Therefore, the slopes of the new asymptotes are

$$\begin{aligned} m &= \pm\frac{1 \times \text{vertical stretch}}{1 \times \text{horizontal stretch}} \\ &= \pm\frac{2}{3} \end{aligned}$$

Because the asymptotes pass through the origin, their equations are of the form $y = mx$. Thus, the asymptotes are $y = \frac{2}{3}x$ and $y = -\frac{2}{3}x$.

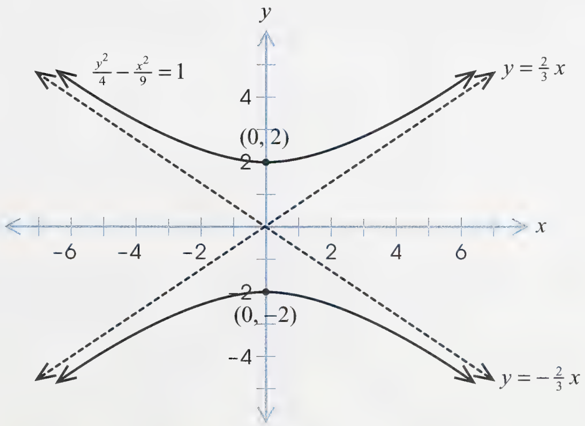
The asymptotes could also be obtained from the original asymptotes, $y = \pm x$, by applying the same transformations the hyperbola underwent.

Substitute $\frac{x}{3}$ for x and $\frac{y}{2}$ for y in $y = \pm x$.

$$\begin{aligned} y &= \pm x \\ \frac{y}{2} &= \pm\frac{x}{3} \\ y &= \pm\frac{2}{3}x \end{aligned}$$

Thus, the new asymptotes are $y = \frac{2}{3}x$ and $y = -\frac{2}{3}x$.

Sketch the graph by first locating the vertices. Then draw the asymptotes using their slopes. Finally, sketch the branches of the hyperbola opening upward and downward from the vertices and between the asymptotes.



Apply these techniques to question 2.

2. For each hyperbola, determine the location of the vertices and the slopes and equations of the asymptotes. Sketch each curve.

a. $\frac{x^2}{25} - y^2 = 1$

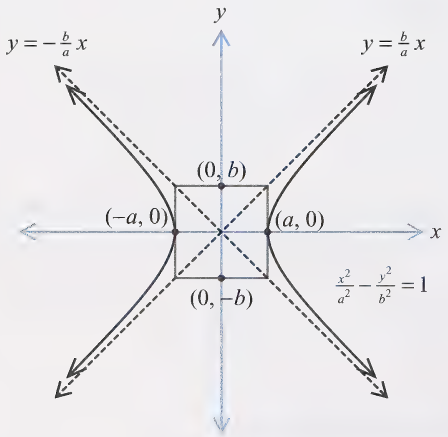
b. $\frac{y^2}{4} - \frac{x^2}{25} = 1$

Turn to page 132 to compare your responses with those in the Appendix.

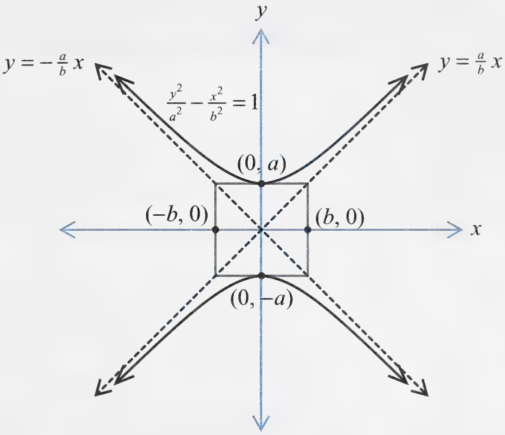
The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the **standard form** of a hyperbola centred at the origin with a transverse axis along the x-axis. The vertices are at $(\pm a, 0)$; the slopes of the asymptotes are $\pm\frac{b}{a}$; and the graph is the graph of $x^2 - y^2 = 1$ stretched horizontally by a factor of a and vertically by a factor of b.

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ☒ Reasoning
- ■ Technology
- ☒ Visualization



The equation $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is the **standard form** of a hyperbola centred at the origin with a transverse axis along the y-axis. The vertices are at $(0, \pm a)$; the slopes of the asymptotes are $\pm\frac{a}{b}$; and the graph is the graph of $y^2 - x^2 = 1$ stretched horizontally by a factor of b and vertically by a factor of a.



If you translate these hyperbolas so they are centred at (h, k), the standard forms of the resulting curves are as follows:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \text{ and } \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☒ Visualization

Example

Sketch the graph of $\frac{(x-3)^2}{25} - \frac{(y-2)^2}{16} = 1$.

Solution

Compare $\frac{(x-3)^2}{25} - \frac{(y-2)^2}{16} = 1$ with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.

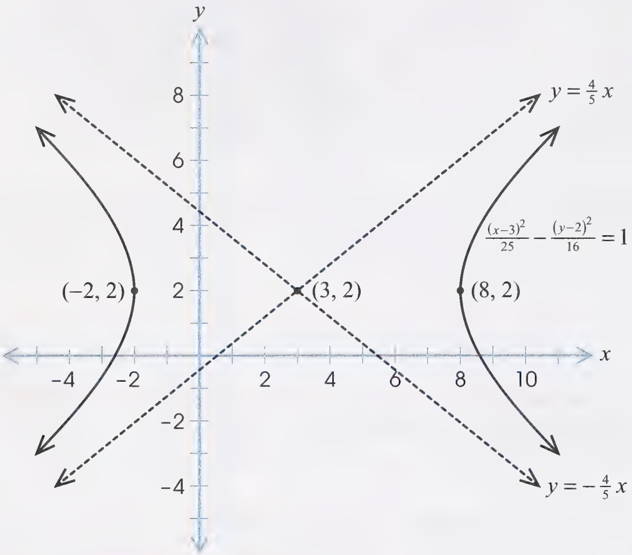
$$\therefore\ h = 3,\ k = 2,\ a = \sqrt{25} = 5, \text{ and } b = \sqrt{16} = 4$$

The centre of the hyperbola is at (3, 2); the transverse axis is parallel to the x-axis; and the hyperbola opens to the left and to the right.

Because $a = 5$, the vertices are 5 units to the left and right of the centre, (3, 2). Therefore, the vertices are at $(3 - 5, 2) = (-2, 2)$ and $(3 + 5, 2) = (8, 2)$.

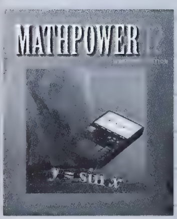
The slopes of the asymptotes are $\pm\frac{b}{a} = \pm\frac{4}{5}$.

Sketch the graph by first locating the centre and the vertices. Then draw the asymptotes from their slopes. Finally, sketch the branches of the hyperbola opening to the left and to the right from the vertices and between the asymptotes.



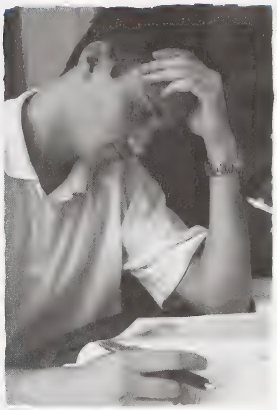
3. Turn to pages 160 and 161 of *MATHPOWER 12* and answer the following.

   a. questions 7, 9, 13, and 31 of "Practice"
   b. question 38 of "Applications and Problem Solving"



Turn to page 133 to compare your responses with those in the Appendix.

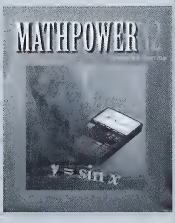
Hyperbolas, like other relations, can be stretched about lines other than the coordinate axes. View the animation *Stretching a Hyperbola About* $y = -3$ from the Pure Mathematics 30 Multimedia CD to see how you could do this by hand. This animation shows the results of stretching $\frac{(x-1)^2}{5^2} - \frac{(y-6)^2}{4^2} = 1$ horizontally by a factor of 2 about the line $x = -3$. Notice how the distance from the line to the new hyperbola is twice the distance to the given hyperbola.



So far, you have analyzed equations written in standard form. Each of these equations can be written in general form, $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

In the next part of this activity you will determine equations of hyperbolas in both standard form and general form from their graphs or descriptions of their graphs.

As you determine these equations, keep track of the coefficients of the equations in general form. What combinations of coefficients will result in hyperbolas?



Turn to page 156 of *MATHPOWER 12* and read from the red line to the bottom of page 157, working through Example 1.

4. Answer the following on pages 160 to 163 of the textbook.

- **a.** questions 14, 16, 19, and 23 of “Practice”
- **b.** questions 44, 45, and 52 of “Applications and Problem Solving”

Turn to page 138 to compare your responses with those in the Appendix.

Mathematical Process

☐ Communication
☒ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☒ Reasoning
☐ Technology
☐ Visualization

From the results of the previous questions, you should be able to recognize the equation of a hyperbola with vertical and horizontal axes of symmetry when the equation is expressed in general form.

In general, given an equation of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where $AC < 0$, the equation represents a hyperbola or one of its degenerate forms.

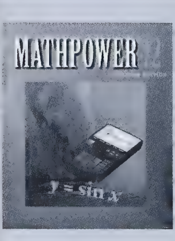
For example, $(x-2)^2 - \frac{(y+1)^2}{9} = 1$ in general form is $9x^2 - y^2 - 36x - 2y + 26 = 0$, where $A = 9$, $B = 0$, and $C = -1$.

If you multiply both sides of this equation by -1, the equation becomes $-9x^2 + y^2 + 36x + 2y - 26 = 0$, where $A = -9$, $B = 0$, and $C = 1$.

In both instances, $AC < 0$ and $B = 0$.

What if you were given an equation of a hyperbola in general form? How would you express it in standard form?

Turn to page 158 of *MATHPOWER 12* and work through Example 2.



5. Answer questions 33 and 35 of "Practice" on page 161 of the textbook.

6. Convert each equation into standard form; then describe and sketch the conic.

   a. $9x^2 - y^2 - 36x - 2y + 26 = 0$
   b. $x^2 - y^2 - 4x - 6y - 5 = 0$

Turn to page 147 to compare your responses with those in the Appendix.

Now Try This

Mathematical Process
- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☒ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

In this activity you converted equations in general form into standard form and vice versa. You discovered that equations in standard form are more useful when analyzing the graphs of hyperbolas.

Numbers, too, have different forms.

Scientific Notation	Exponential Form	Standard Form
1.28×10^2	2^7	128

In the next problem you will work with exponential form and standard form and investigate resulting patterns.

Turn to "PROBLEM SOLVING" on pages 136 and 137 of *MATHPOWER 12* and read "Look for a Pattern."

7. Answer question 4 of "Applications and Problem Solving" on page 137 of the textbook.

Turn to page 152 to compare your response with the one in the Appendix.

Looking Back

Mathematical Process

☒ Communication
■ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
■ Technology
■ Visualization

In this activity you explored the hyperbola and its graph by starting from $x^2 - y^2 = 1$ or $y^2 - x^2 = 1$. You discovered that you can obtain any hyperbola with vertical and horizontal axes by transforming the graph of $x^2 - y^2 = 1$ or $y^2 - x^2 = 1$. These transformations led to the standard form of the hyperbola. From its standard form, you identified the critical features of the hyperbola and used those features to sketch its graph. You also examined the general form of the equation, and you converted equations from general form into standard form and vice versa.

In your journal, compare the general form of an ellipse with the general form of a hyperbola. How are they similar? How do they differ? You may wish to illustrate your explanation with examples.

Activity 2: The Parabola

parabola: the conic produced by slicing a right-circular cone parallel to one of its generators

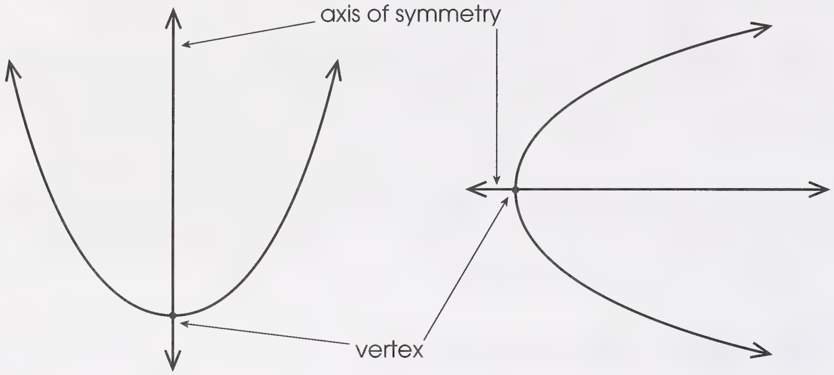
Mathematical Process

■ Communication
☒ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
■ Technology
■ Visualization

Did you know that engineers and architects incorporate the **parabola** in many of their designs? Because of their load-bearing characteristics, suspension bridges and bridge spans are parabolic. Parabolic surfaces can also be found in solar furnaces, radio telescopes, satellite antennas, vehicle headlamps, and flashlights. The parabola can even be found in the designs of some golf clubs.

In this activity you will explore the parabola and its equation. You have already studied the parabola in detail in previous mathematics courses. In Pure Mathematics 20 you examined quadratic functions, their application, and their graphs. In Module 1 of this course you looked at the transformation of the basic quadratic function $y = x^2$. You may wish to review these transformations before continuing with this activity.

Recall that a parabola has only one axis of symmetry and that the **vertex** of the parabola appears where the axis of symmetry and the parabola intersect. In this activity you will restrict your investigation to parabolas with a vertical or horizontal axis of symmetry.



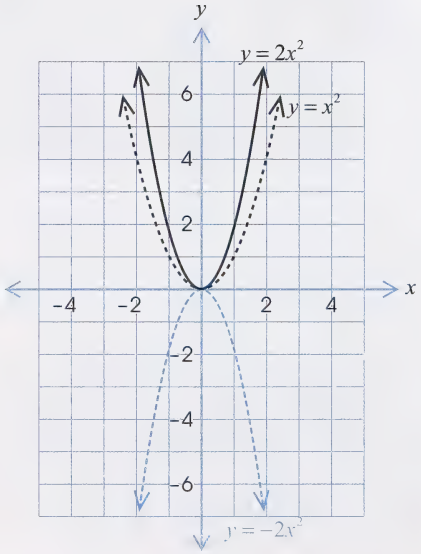
Example

Describe the relationships among the graphs $y = x^2$, $y = 2x^2$, $y = -2x^2$, $x = 2y^2$, and $x = -2y^2$.

Solution

The graph of $y = 2x^2$ or $\frac{y}{2} = x^2$ is the graph of $y = x^2$ stretched vertically by a factor of 2. This is consistent with the ellipse and hyperbola. To stretch the curves of either of these conics vertically by a factor of 2, you would replace y with $\frac{y}{2}$.

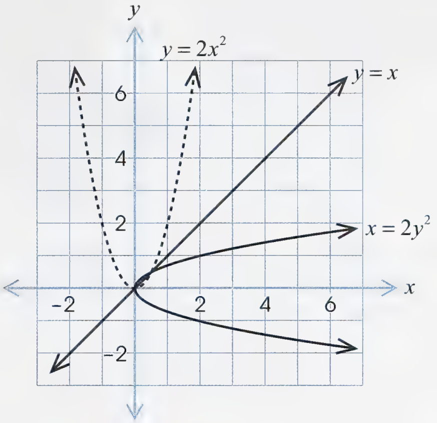
The graph of $y = -2x^2$ is the graph of $y = 2x^2$ reflected in the x-axis. All three graphs ($y = x^2$, $y = 2x^2$, and $y = -2x^2$) have their axis of symmetry along the y-axis and their vertex at the origin.



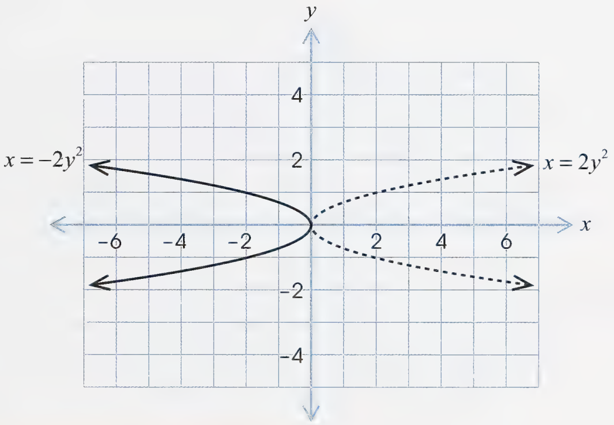
Mathematical Process

- ■ Communication
- ☒ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ☒ Visualization

Because $y = 2x^2$ and $x = 2y^2$ are inverses—that is, the variables have been interchanged—the graph of $x = 2y^2$ is the graph of $y = 2x^2$ reflected in the line $y = x$. Therefore, the axis of symmetry of $x = 2y^2$ lies along the x-axis and the vertex is at the origin.



The graph of $x = -2y^2$ is the graph of $x = 2y^2$ reflected in the y-axis.

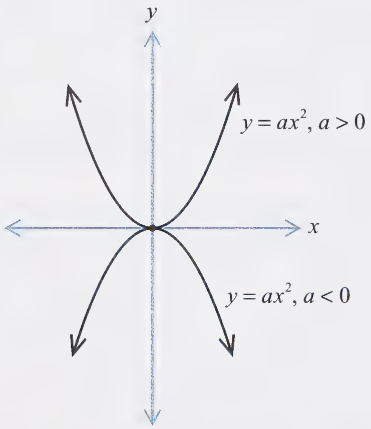


The results of the previous example can be summarized as follows.

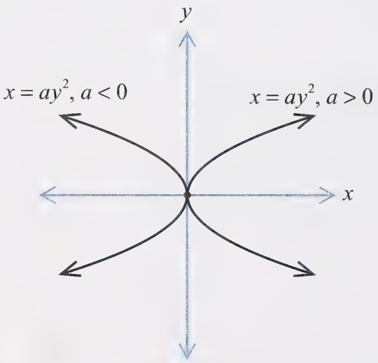
Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ☒ Reasoning
- ■ Technology
- ☒ Visualization

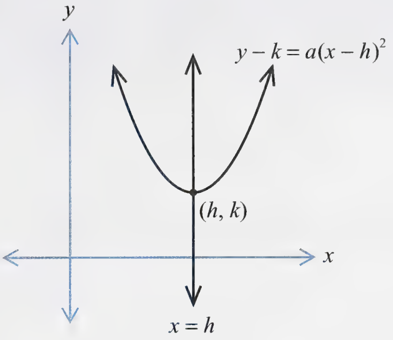
The **standard form** of a parabola with its vertex at the origin and its axis of symmetry along the y-axis is $y = ax^2$. If $a > 0$, the graph opens upward. If $a < 0$, the graph opens downward.

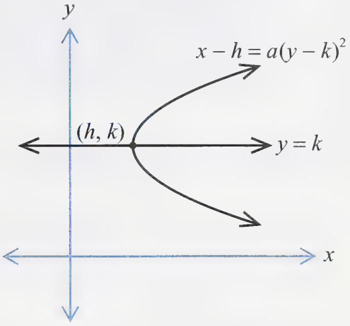


The **standard form** of a parabola with its vertex at the origin and its axis of symmetry along the x-axis is $x = ay^2$. If $a > 0$, the graph opens to the right. If $a < 0$, the graph opens to the left.



If the vertex is moved off the origin to point (h, k), the resulting equations are $y - k = a(x - h)^2$ and $x - h = a(y - k)^2$.





Mathematical Process

☐ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☒ Reasoning
☐ Technology
☒ Visualization

Example

a. Describe how the graph of $y + 4 = 2(x - 3)^2$ is related to the graph of $y = x^2$. Draw both graphs.

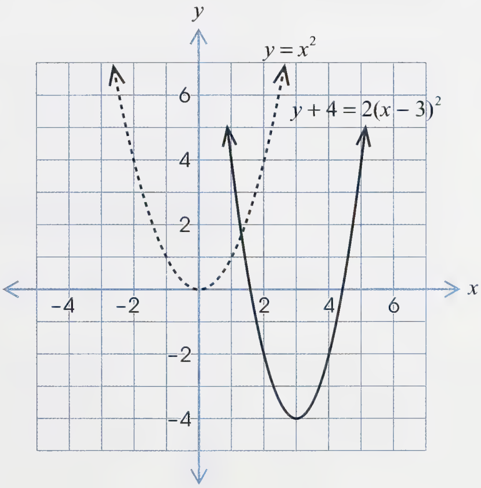
b. For each parabola, state the vertex, axis of symmetry, domain, and range.

Solution

a. Compare $y + 4 = 2(x - 3)^2$ with $y - k = a(x - h)^2$.

$\therefore a = 2, h = 3, \text{ and } k = -4$

The graph of $y + 4 = 2(x - 3)^2$ is the graph of $y = x^2$ stretched vertically by a factor of 2 and then translated 3 units to the right and 4 units down.



b.

Graph	Vertex	Axis of Symmetry	Domain	Range
$y = x^2$	(0, 0)	$x = 0$	Reals	$y \geq 0$
$y + 4 = 2(x - 3)^2$	(3, −4)	$x = 3$	Reals	$y \geq -4$

1. Identify the vertex and axis of symmetry for each parabola; then sketch the graph. State the domain and range for each graph.

   a. $y + 2 = -2(x - 1)^2$
   b. $x + 2 = (y - 4)^2$
   c. $x - 3 = -2(y + 1)^2$

Turn to page 152 to compare your responses with those in the Appendix.

You have just graphed parabolas from their equations. Next, you will determine the equations of parabolas from their graphs. As for circles, ellipses, and hyperbolas, you can express these equations in standard form or general form.

Example

In standard form and general form, determine the equation of a parabola with its vertex at (2, 3) with a horizontal axis of symmetry, and passing through (10, 5). Graph the parabola.

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ☒ Problem Solving
- ☒ Reasoning
- ■ Technology
- ☒ Visualization

Solution

Because the axis of symmetry is horizontal, the standard form of the equation of this parabola is $x - h = a(y - k)^2$.

The vertex is (2, 3); so, $h = 2$ and $k = 3$.

$$\begin{aligned} \therefore\ x - h &= a(y - k)^2 \\ x - 2 &= a(y - 3)^2 \end{aligned}$$

Next, find a. Because the graph passes through (10, 5), substitute $x = 10$ and $y = 5$ into the equation.

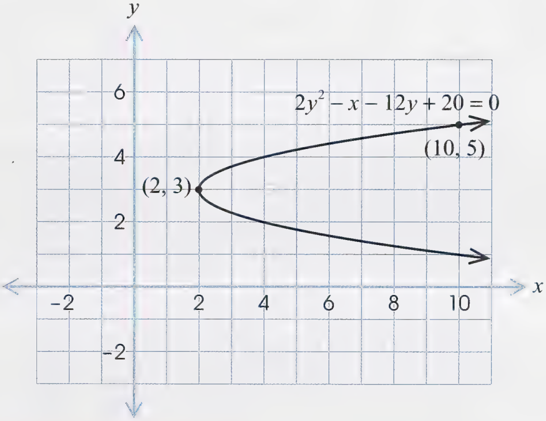
$$\begin{aligned} x-2&=a(y-3)^2\\ 10-2&=a(5-3)^2\\ 8&=a(2)^2\\ 8&=4a\\ a&=2 \end{aligned}$$

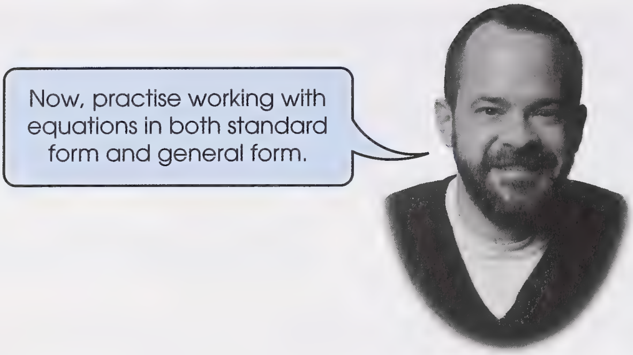
The standard form of the equation is $x - 2 = 2(y - 3)^2$.

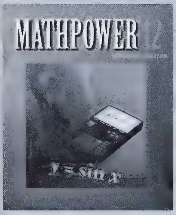
Convert the equation into general form.

$$\begin{aligned} x-2&=2(y-3)^2\\ x-2&=2(y^2-6y+9) \quad \leftarrow \text{Expand.}\\ x-2&=2y^2-12y+18\\ 2y^2-x-12y+20&=0 \end{aligned}$$

The general form of the equation is $2y^2 - x - 12y + 20 = 0$.




Now, practise working with equations in both standard form and general form.



2. Turn to pages 168 and 169 of *MATHPOWER 12* and answer the following.

- **a.** questions 12 and 25 of "Practice"
- **b.** questions 47 and 49.c. of "Applications and Problem Solving"

Turn to page 154 to compare your responses with those in the Appendix.

You may have noticed that when $x - h = a(y - k)^2$ is written in general form, the result is $Cy^2 + Dx + Ey + F = 0$. Also, when $y - k = a(x - h)^2$ is written in general form, the result is $Ax^2 + Dx + Ey + F = 0$.

In general, given an equation of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where $AC = 0$, the equation represents a parabola or one of its degenerate forms.

If you are given an equation in general form, you should be able to convert it to standard form and sketch the curve. **Note:** As for the other conics, the standard form is also called completed-square form.

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ☒ Reasoning
- ■ Technology
- ☒ Visualization

Example

Express $x^2 - 4x + 2y - 8 = 0$ in standard form. Then state the vertex, axis of symmetry, and direction of opening; and sketch the graph.

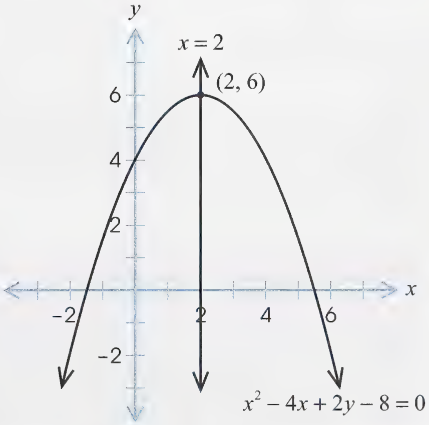
Solution

$$
\begin{aligned}
x^2 - 4x + 2y - 8 &= 0\\
2y - 8 &= -x^2 + 4x\\
2y - 8 &= -\left(x^2 - 4x\right)\\
2y - 8 &= -\left[\left(x^2 - 4x + 4\right) - 4\right]\\
2y - 8 &= -\left[(x-2)^2 - 4\right]\\
2y - 8 &= -(x-2)^2 + 4\\
2y - 12 &= -(x-2)^2\\
2(y-6) &= -(x-2)^2\\
y - 6 &= -\frac{1}{2}(x-2)^2
\end{aligned}
$$

Compare $y - 6 = -\frac{1}{2}(x-2)^2$ with $y - k = a(x-h)^2$.

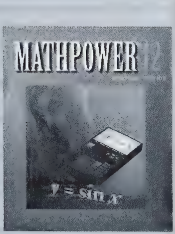
$\therefore\ a = -\frac{1}{2},\ h = 2,$ and $k = 6$

The vertex is at (2, 6); the axis of symmetry is $x = 2$; and the parabola opens downward.



Remember: Always end up with one of these forms when completing the square:

$$y - k = a(x-h)^2 \text{ or } x - h = a(y-k)^2$$



3. Turn to page 168 of *MATHPOWER 12* and answer questions 36, 37, and 39 of "Practice."

Turn to page 158 to compare your responses with those in the Appendix.

Parabolas, like other relations, can be stretched about lines other than the coordinate axes. View the animation *Stretching a Parabola About* $y = -4$ from the Pure Mathematics 30 Multimedia CD to see how you could do this by hand. This animation shows the results of stretching $3x^2 + 10y - 80 = 0$ vertically by a factor of $\frac{1}{3}$ about the line $y = -4$. Notice how the distance from the line to the new parabola is $\frac{1}{3}$ the distance to the given parabola.

This stretch can also be done algebraically. Just remember to translate the equation so the stretch line matches an axis, and stretch the new equation. Then translate the result so the stretch line is back in its original place.

4. Detemine the general form of the equation of the stretched parabola in the animation.

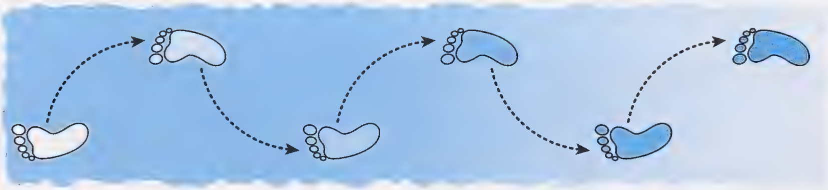
Turn to page 161 to compare your responses with the ones in the Appendix.

Now Try This

The process of converting from general form to standard form by completing the square is similar to working backward. After all, the general form was obtained in the first place by expanding the equation in standard form and rearranging the terms.

Mathematical Process

- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☒ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization



Working backward is an important problem-solving skill. Try the following problem. It may best be approached by working backward.

Turn to “PROBLEM SOLVING” on pages 172 and 173 of *MATHPOWER 12* and read “Work Backward.”

5. Answer question 9 of “Applications and Problem Solving” on page 173 of the textbook.

Turn to page 163 to compare your response with the one in the Appendix.

Mathematical Process

- ☒ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

Looking Back

In this activity you explored the equations and graphs of parabolas and converted their equations from standard form to general forms and vice versa.

Now that you have covered all four conics, how would you explain the restrictions on the coefficients of the general form, $Ax^2 + Cy^2 + Dx + Ey + F = 0$, required to produce a circle? an ellipse? a hyperbola? a parabola? Record your explanations in your journal.

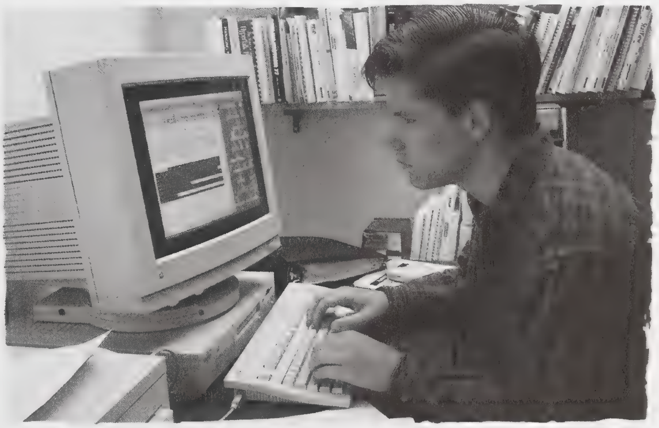
Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

When you were exploring the hyperbola, did you have any difficulty converting between general form and standard form or using standard form to sketch and analyze these curves?

To reinforce these procedures, open *The Quadratic Relations Explorer* in the Explorers folder from the Pure Mathematics 30 Companion CD. **Remember:** The graphics and procedures that follow are for the Windows version.

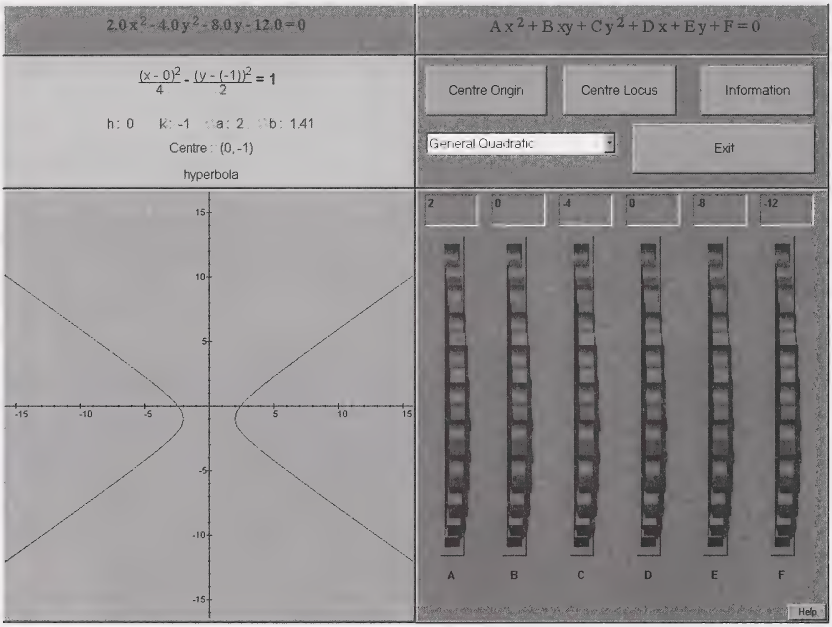


The initial screen of *The Quadratics Relations Explorer* allows you to alter the coefficients of the general form, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. You can either change the values of the coefficients by clicking and dragging the wheels or by simply clicking in the rectangles above the wheels and entering the desired values. Once you enter a value for a coefficient, the graph will automatically change.

For example, enter the following equation for a horizontal hyperbola:

$$2x^2 - 4y^2 - 8y - 12 = 0$$

To convert the general form of the hyperbola to standard form, right-click the rectangle above the graph with your mouse. The following display should appear.



$2.0x^2-4.0y^2-8.0y-12.0=0$

$\frac{(x-0)^2}{4} - \frac{(y-(-1))^2}{2} = 1$

h: 0 k: -1 a: 2 b: 1.41

Centre : (0,-1)

hyperbola

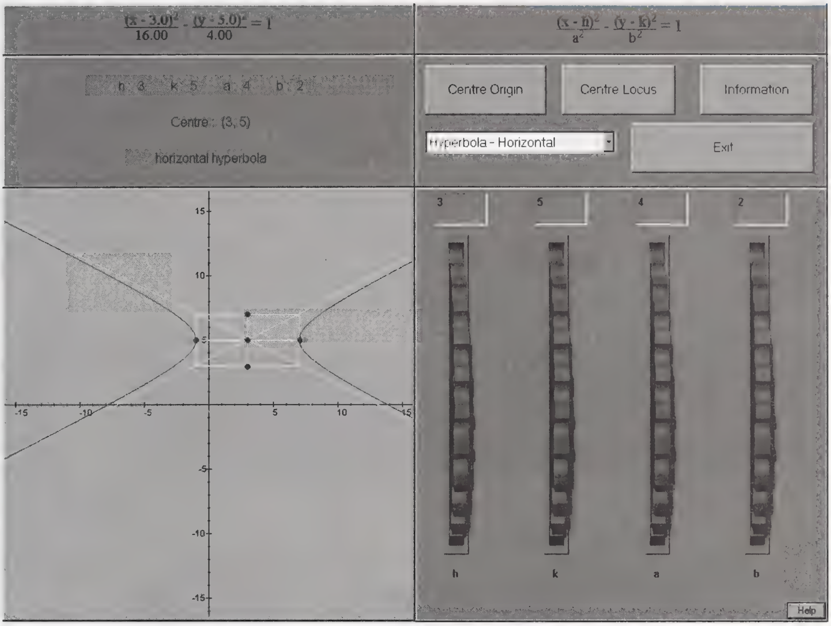
$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

General Quadratic

A: 2
B: 0
C: -4
D: 0
E: -8
F: -12

You can also enter the equation of a hyperbola given in standard form. Change the pull-down menu where it reads “General Quadratic” so it now reads “Hyperbola-Horizontal” or “Hyperbola-Vertical.” If you select a hyperbola with a horizontal transverse axis, for example, you can enter the values of *h*, *k*, *a*, and *b* in the same way as the coefficients for the general form.

The following display shows an example of an equation of a horizontal hyperbola in standard form.



$$\frac{(x-3.0)^2}{16.00}-\frac{(y-5.0)^2}{4.00}=1$$

h 3 k 5 a 4 b 2

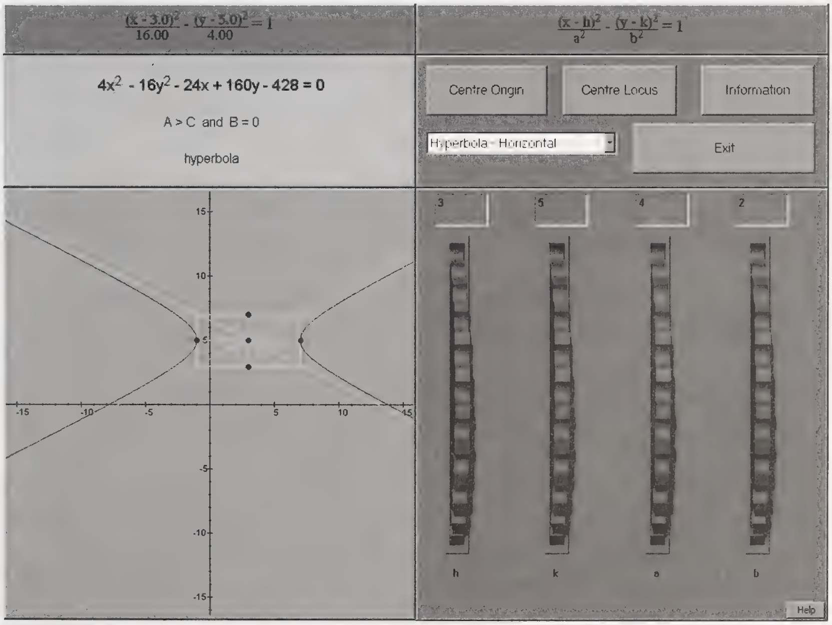
Centre: (3, 5)

horizontal hyperbola

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Hyperbola - Horizontal

You can also change the form of this hyperbola from standard form to general form by right-clicking the rectangle above the graph. The following display should appear.



$$\frac{(x-3.0)^2}{16.00}-\frac{(y-5.0)^2}{4.00}=1$$

$4x^2 - 16y^2 - 24x + 160y - 428 = 0$

A > C and B = 0

hyperbola

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Hyperbola - Horizontal

Notice that when you use *The Quadratic Relations Explorer* to convert an equation from standard form into general form, the result often can be reduced by dividing by a common factor. In the preceding case the answer given was $4x^2 - 16y^2 - 24x + 160y - 428 = 0$. Because each coefficient is a multiple of 4, you can divide each term by 4 to reduce the equation.

$$\frac{4x^2}{4} - \frac{16y^2}{4} - \frac{24x}{4} + \frac{160y}{4} - \frac{428}{4} = \frac{0}{4}$$
$$x^2 - 4y^2 - 6x + 40y - 107 = 0$$

Remember: There are an infinite number of ways of writing the equation of a particular hyperbola. Simply multiply or divide each term in the equation by a non-zero constant.

Take a few minutes to experiment with the graphs and equations of hyperbolas and parabolas. Then use *The Quadratic Relations Explorer* to check your answers to the following questions.

1. What are the domain and range of the hyperbola $\frac{(x+1)^2}{4} - \frac{(y-2)^2}{9} = 1$? Express the equation in general form.
2. Convert $16x^2 - 9y^2 - 32x - 72y + 16 = 0$ to standard form. What are the domain and range of this hyperbola?
3. What is the equation of the hyperbola with centre (2, 3), a horizontal transverse axis 6 units in length, and asymptotes with slope $\pm\frac{1}{2}$? Write your answer in both standard form and general form.

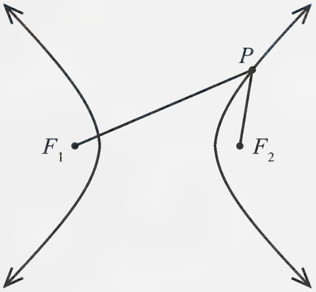
Turn to page 163 to compare your responses with those in the Appendix.

Enrichment

In Section 2 you derived the standard form of the equation of a hyperbola by transforming the rectangular hyperbola $x^2 - y^2 = 1$. You can obtain the same result by using the **locus definition** of a hyperbola. The locus definition tells you how the points on a hyperbola are located relative to two fixed points, called foci.

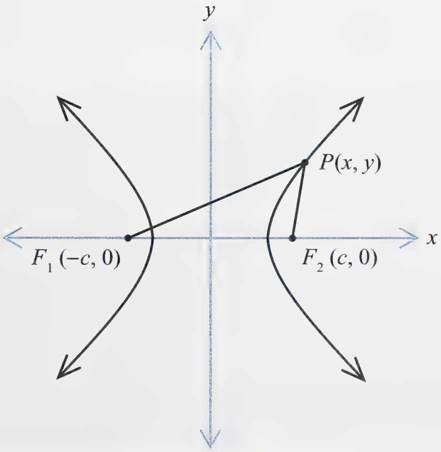
Locus Definition

A hyperbola is a set of points such that the absolute value of the difference between the distances each point lies from two fixed points is constant.



In the diagram, F_1 and F_2 are the foci of the hyperbola and $|PF_1 - PF_2|$ is a constant, regardless of the position of point P on the curve.

For example, suppose F_1 and F_2 are located at $(-c, 0)$ and $(c, 0)$ on the x-axis (as shown in the diagram) and $|PF_1 - PF_2| = 2a$. What is the equation of this hyperbola?



Because $|PF_1 - PF_2| = 2a$,

$$\left|\sqrt{\left[x-(-c)\right]^2+(y-0)^2}-\sqrt{(x-c^2)+(y-0)^2}\right|=2a$$

$$\left|\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}\right|=2a$$

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a$$

$$\sqrt{(x+c)^2+y^2}=\pm 2a+\sqrt{(x-c)^2+y^2}$$

$$\left(\sqrt{(x+c)^2+y^2}\right)^2=\left(\pm 2a+\sqrt{(x-c)^2+y^2}\right)^2$$

$$(x+c)^2+\cancel{y}^2=4a^2\pm 4a\sqrt{(x-c)^2+y^2}+(x-c)^2+\cancel{y}^2$$

$$(x+c)^2=4a^2\pm 4a\sqrt{(x-c)^2+y^2}+(x-c)^2$$

$$\cancel{x}^2+2cx+\cancel{c}^2=4a^2\pm 4a\sqrt{(x-c)^2+y^2}+\cancel{x}^2-2cx+\cancel{c}^2$$

$$\cancel{4}cx=\cancel{4}a^2\pm\cancel{4}a\sqrt{(x-c)^2+y^2}$$

$$cx=a^2\pm a\sqrt{(x-c)^2+y^2}$$

$$cx-a^2=\pm a\sqrt{\left(x-c^2+y^2\right)}$$

$$\left(cx-a^2\right)^2=\left(\pm a\sqrt{(x-c)^2+y^2}\right)^2$$

$$c^2x^2-2a^2cx+a^4=a^2\left[(x-c)^2+y^2\right]$$

$$c^2x^2-2a^2cx+a^4=a^2\left(x^2-2cx+c^2+y^2\right)$$

$$c^2x^2-\cancel{2a^2cx}+a^4=a^2x^2-\cancel{2a^2cx}+a^2c^2+a^2y^2$$

$$c^2x^2-a^2x^2-a^2y^2=a^2c^2-a^4$$

$$x^2\left(c^2-a^2\right)-a^2y^2=a^2\left(c^2-a^2\right)$$

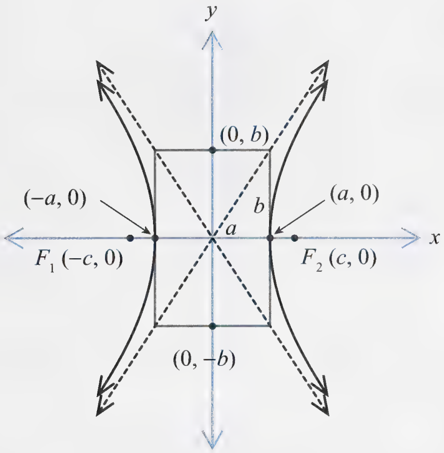
Now substitute b^2 for (c^2-a^2).

$$b^2x^2-a^2y^2=a^2b^2$$

$$\frac{\cancel{b}^2x^2}{a^2\cancel{b}^2}-\frac{\cancel{a}^2y^2}{\cancel{a}^2b^2}=\frac{a^2b^2}{a^2b^2}$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

This is the equation of a hyperbola centred at the origin that crosses the x-axis at $(\pm a, 0)$ in standard form. The asymptotes of this hyperbola are of slope $\pm\frac{b}{a}$.



The relationship $b^2 = c^2 - a^2$ or $c^2 = a^2 + b^2$ provides a relationship among three parameters. The parameter c is the distance the foci lie from the centre; the parameter a is the distance the vertices lie from the centre; and the parameter b is used to determine the slope, $\pm\frac{b}{a}$, of the asymptotes.

Now, try your hand at the following.

1. Find the equation of the hyperbola with foci at (0, 5) and (0, −5) if the absolute value of the difference of the distances each point on the hyperbola lies from the foci is 8 units.

2. Determine the foci of $\frac{x^2}{144} - \frac{y^2}{25} = 1$.

Turn to page 168 to compare your responses with those in the Appendix.

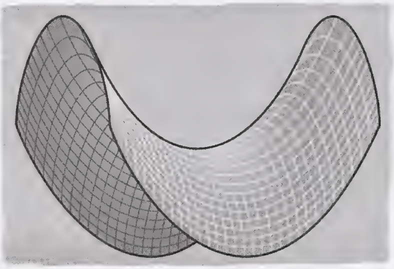
Conclusion

In this section you explored hyperbolas and parabolas, their graphs, and their equations.

For hyperbolas, you restricted your discussion to curves with horizontal and vertical axes of symmetry. You began deriving the standard form of their equations by transforming the rectangular hyperbola $x^2 - y^2 = 1$ or $y^2 - x^2 = 1$. You then used the standard form to sketch and describe these hyperbolas.

For parabolas with a horizontal or vertical axis of symmetry, you derived their standard form by transforming $y = x^2$. You then used the standard form to sketch and describe these parabolas.

For both the hyperbola and parabola, you discovered how to recognize these curves from the coefficients of an equation in general form. You also converted the equations from standard form into general form and vice versa.



The shape of a saddle is easily recognized. This shape resembles a hyperbolic paraboloid, which incorporates both the hyperbola and the parabola. Architects have incorporated the hyperbolic paraboloid in such structures as the Saddledome in Calgary, the Georgia Dome in Atlanta, the Oakland International Airport, and St. Mary's Cathedral in San Francisco.

Assignment

Turn to Assignment Booklet 4B and complete the assignment for Section 2.

Module Summary

In this module you explored the conic sections—the circle, the ellipse, the hyperbola, and the parabola—and their degenerate forms.

In Section 1 you investigated the curves that can be obtained by slicing a double-napped, right-circular cone and a cylinder. In particular, you looked at the conics in relation to the orientation of the intersecting plane. You then explored two of these conics—the circle and ellipse—in detail. You analyzed their graphs and equations and used transformations to change their shape, orientation, and location in the coordinate plane.

In Section 2 you explored the hyperbola and the parabola using transformations, once again, to analyze their graphs and derive their equations.

Throughout the module you worked with the standard form and the general form of the equations of the conic sections. You discovered how to identify curves from their equations and practised converting equations from one form into the other.

Much of the geometry you study is a generalization of the patterns and structures observed in nature. It is a human characteristic to see patterns in the everyday world. These patterns are studied and lead to new discoveries in mathematics. You can thank the Greek geometers for their recognition that the circle, the ellipse, the parabola, and the hyperbola are sections of the cone.

Final Module Assignment

Turn to Assignment Booklet 4B and complete the final module assignment. Remember to submit your completed Assignment Booklet for assessment.

Notes

Appendix

Glossary
Calculator Functions
Suggested Answers
Credits

Glossary

axis of symmetry: a line in which the two halves of a curve reflect into each other

central axis: the axis of symmetry of a right-circular cone

centre: the intersection of the axes of symmetry in an ellipse or hyperbola

circle: the conic produced by slicing a right-circular cone at right angles to its central axis

conic section: a curve produced when a plane intersects a right-circular cone

conjugate axis: the axis of symmetry perpendicular to the transverse axis of a hyperbola

degenerate conic section: a point, line, or pair of lines that arise as a limiting form of a conic

double-napped cone: two identical but opposite cones that share a common vertex

ellipse: a conic produced by stretching or compressing a circle along an axis of symmetry

general form: a second-degree equation written as $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where at least one of A, B, and C is a non-zero

generator: a line that sweeps out a surface, such as a cylinder or a right-circular cone, when rotated

hyperbola: the conic produced by slicing a right-circular cone parallel to its axis or at an angle to the axis less than the angle of the generator

major axis: the longer axis of symmetry in an ellipse

minor axis: the shorter axis of symmetry in an ellipse

parabola: the conic produced by slicing a right-circular cone parallel to one of its generators

point circle: a circle of radius zero

right-circular cone: a cone with a circular base and a central axis that meets the base at right angles

standard form: completed-square form

For example, the standard form of a circle is $(x-h)^2 + (y-k)^2 = r^2$.

standard form of a hyperbola: the equation of a hyperbola written in the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

standard form of an ellipse: the equation of an ellipse written in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

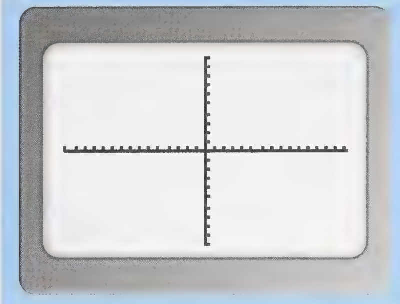
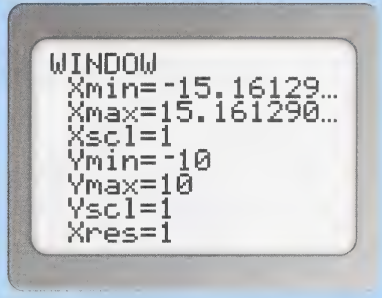
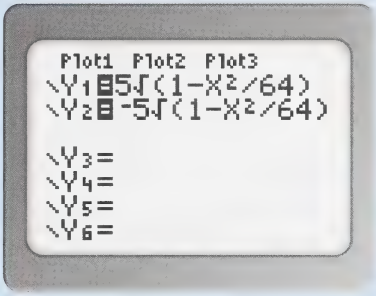
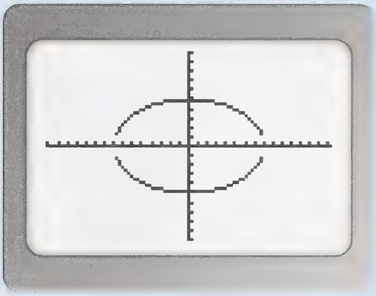
standard form of a parabola: the equation of a parabola written in the form $y - k = a(x-h)^2$ or $x - h = a(y-k)^2$

transverse axis: the axis of symmetry joining the vertices of a hyperbola

vertices of a hyperbola: the endpoints of the transverse axis

vertices of an ellipse: the endpoints of the major axis

Calculator Functions

Calculator Function	Displays
Obtaining a Square Display ZOOM 5 This display is used so graphs do not appear distorted.	  WINDOW Xmin=-15.16129… Xmax=15.161290… Xscl=1 Ymin=-10 Ymax=10 Yscl=1 Xres=1
Graphing an Ellipse or Circle e.g., $\frac{x^2}{64}+\frac{y^2}{25}=1$ **1:** Solve for y. $y=\pm5\sqrt{1-\frac{x^2}{64}}$ **2:** Enter the values of y into the equation editor. **3:** Press GRAPH to graph the ellipse. **Note:** Always graph an ellipse or circle on a square display.	 Plot1 Plot2 Plot3 \Y1=5√(1-X²/64) \Y2=-5√(1-X²/64) \Y3= \Y4= \Y5= \Y6= 

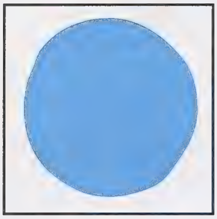
Suggested Answers

Section 1: Activity 1

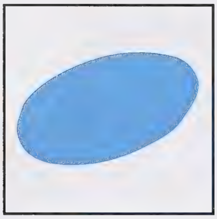
1. **a. Textbook questions a. to d. of "Explore 2: Make a Model," p. 134**

Answers may vary. A sample print is given.

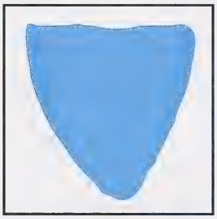
a.



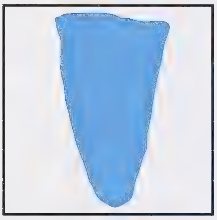
b.



c.



d.



b. Textbook questions 1 to 3 of "Inquire," p. 135

1. **a.** circle **b.** ellipse **c.** hyperbola **d.** parabola

2. A hyperbola can be generated only by a double-napped cone.

3. Answers may vary. A sample answer is given.

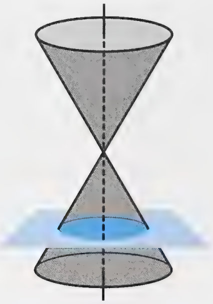
A circle is a set of points equidistant from a fixed point. There is an infinite number of axes of symmetry.

An ellipse is a circle stretched vertically or horizontally. An ellipse has two axes of symmetry.

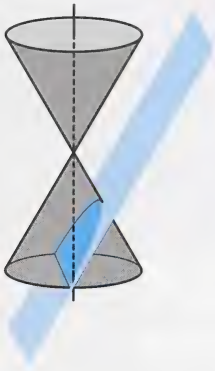
A hyperbola has two branches and two axes of symmetry.

A parabola consists of a single branch and one axis of symmetry.

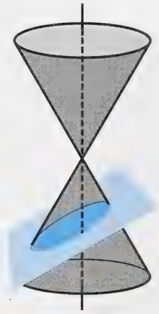
2. **a.** A circle is formed when the intersecting plane crosses the central axis at 90°.



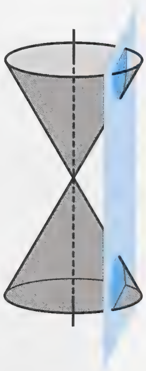
b. A parabola is formed when the intersecting plane crosses the central axis at 20°.



c. An ellipse is formed when the intersecting plane crosses the central axis at any angle, θ, where $20^\circ < \theta < 90^\circ$.

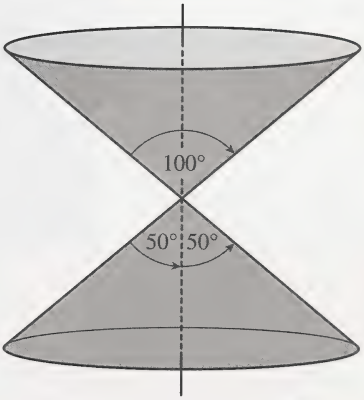


d. A hyperbola is formed when the intersecting plane crosses the central axis at any angle θ, where $0 \le \theta < 20^\circ$.



3. The angle between a generator of the cone and the central axis is

$$\begin{aligned}\frac{1}{2}\times(\text{angle at the vertex}) &= \frac{1}{2}\times 100^\circ \\ &= 50^\circ\end{aligned}$$



Let θ be the angle between the intersecting plane and the central axis of the double-napped cone.

a. $\theta = 90^\circ$

b. $\theta = 50^\circ$

c. $50^\circ < \theta < 90^\circ$

d. $0 \le \theta < 50^\circ$

Section 1: Activity 1 (continued)

4.

Conic	Degenerate Conic
Circle	point or no graph
Ellipse	point or no graph
Parabola	line, two parallel lines, or no graph
Hyperbola	two intersecting lines

5. **a. Textbook questions 1 to 3 of "Practice," p. 135**

   1. The parts of the cone are the vertex, generator, axis, and nappe. The generator is the line that is rotated about the axis to sweepout the cone.

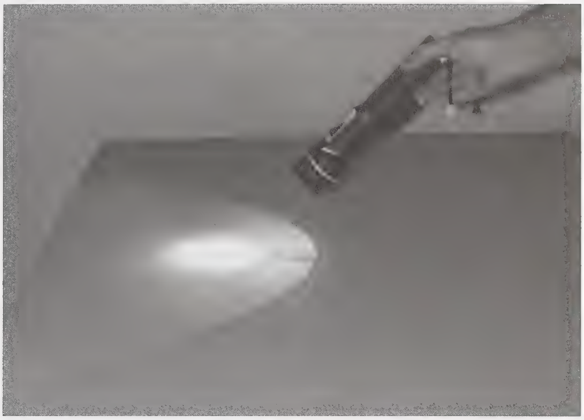
   2. A double-napped cone consists of two opposite and congruent cones that share a common vertex.

   3. A right circular cone has a circular base and an axis that intersects the base at a right angle.

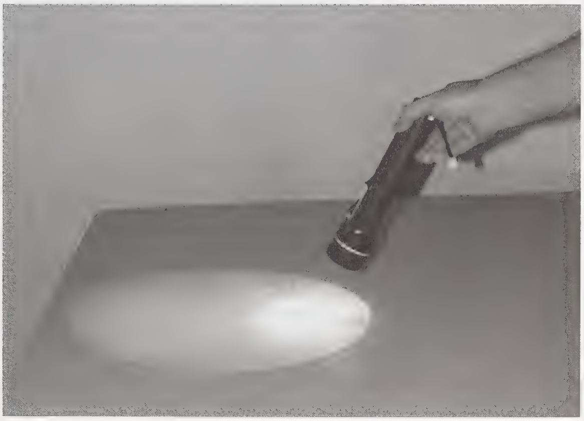
   b. Textbook questions 4, 5, and 9 of "Applications and Problem Solving," p. 135

   4. To form a circle, the flashlight must be held vertically. The tabletop must intersect the axis of the cone of light at 90°.

To form a parabola, tilt the flashlight from the vertical until the uppermost portion of the cone of light is parallel to the tabletop.

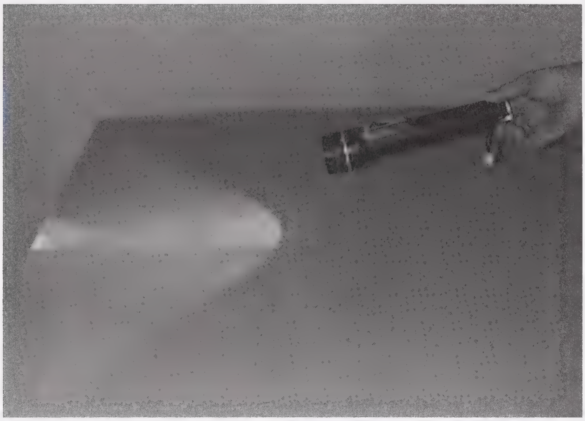


To form an ellipse, tilt the flashlight back toward the vertical; however, the axis of the cone of light must not strike the tabletop at 90°.



Section 1: Activity 1 (continued)

To form a hyperbola, tilt the flashlight toward the horizontal, beyond the angle required to form a parabola.



Note: This photo shows one branch of the hyperbola.

5. The base of the cone is parallel to the plane when a circle is generated.

A generator of the cone is parallel to the plane when a parabola is generated.

The axis of the cone is parallel to the plane when a hyperbola is generated. **Note:** The axis of symmetry does not have to be parallel to the plane to produce a hyperbola.

In the case of the ellipse, no part of the cone is parallel to the plane.

9. **a.** A circle is formed when the jet is flying straight up.
b. A hyperbola or parabola is formed when the jet is climbing gently.
c. An ellipse is formed when the jet is climbing at a steep angle.

6. Textbook questions 21 to 27 of "Mental Math," p. 133

21. 36

22. 16

23. $\frac{9}{4}$

24. 9

25. $\frac{25}{4}$

26. 64

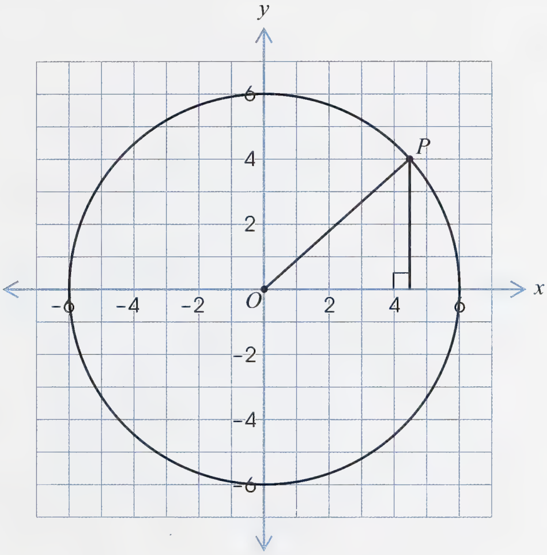
27. $\frac{1}{4}$

Section 1: Activity 2

1. a. **Textbook question "Explore 1: Use a Model," p. 138**

The coordinates of *P* are approximately (4.5, 4.0).

The distance formula gives you the length of the hypotenuse of a right triangle with a base of length $|x_2 - x_1|$ and an altitude of length $|y_2 - y_1|$.



b. **Textbook questions 1 to 4 of "Inquire," p. 138**

1. The radius of the circle is *OP*.

Let $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) \doteq (4.5, 4)$.

$$\begin{aligned}\therefore\ OP &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\\ &\doteq \sqrt{(4.5 - 0)^2 + (4 - 0)^2}\\ &\doteq \sqrt{4.5^2 + 4^2}\\ &\doteq \sqrt{36.25}\\ &\doteq 6.0\end{aligned}$$

The radius is approximately 6.0 units.

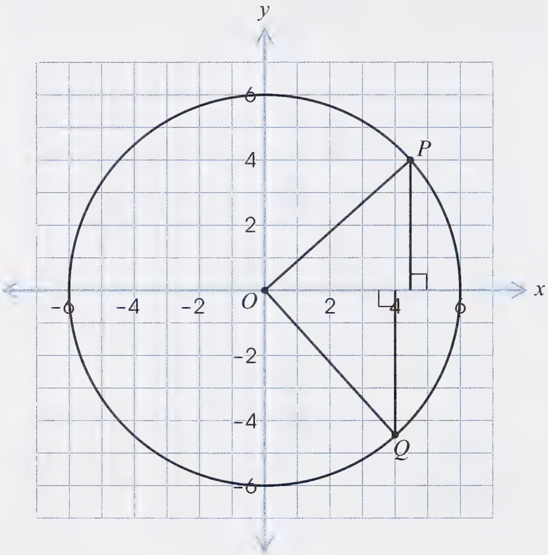
Section 1: Activity 2 (continued)

2. The measured radius and calculated radius are equal.

3. Point Q, in the fourth quadrant, is approximately (4.0, –4.5).

Let $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) \doteq (4, -4.5)$.

$$\begin{aligned} \therefore\ OQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &\doteq \sqrt{(4-0)^2 + (-4.5-0)^2} \\ &\doteq \sqrt{4^2 + (-4.5)^2} \\ &\doteq \sqrt{36.25} \\ &\doteq 6.0 \end{aligned}$$

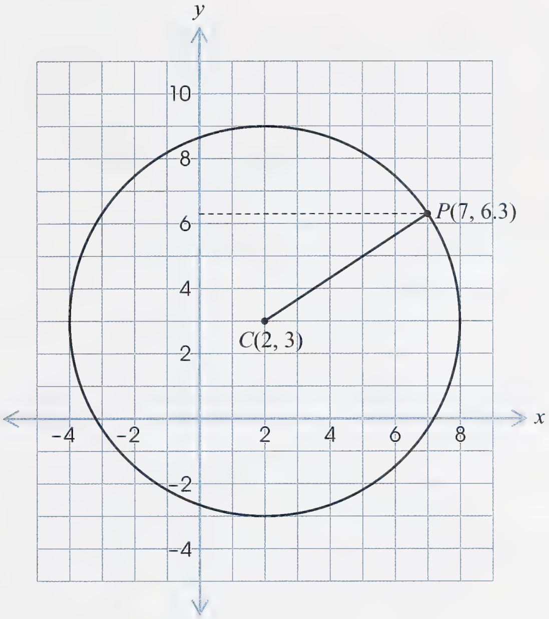


Again, the calculated radius equals the measured radius.

4. $r = \sqrt{(x-0)^2 + (y-0)^2}$ or $x^2 + y^2 = r^2$

$$= \sqrt{x^2 + y^2}$$

c. Textbook question "Explore 2: Use a Model," p. 138



The coordinates of P are approximately (7, 6.3).

d. Textbook questions 1, 2, and 4 of "Inquire," p. 139

1. The radius of the circle did not change when the CD was moved. The radius is 6.0.

2. Let $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) \doteq (7, 6.3)$.

$$\begin{aligned}\therefore\ CP &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\\ &\doteq \sqrt{(7-2)^2 + (6.3-3)^2}\\ &\doteq \sqrt{5^2 + 3.3^2}\\ &\doteq \sqrt{35.89}\\ &\doteq 6.0\end{aligned}$$

This value of the radius is the same as the value from question 1.

Section 1: Activity 2 (continued)

4. Let $(x_1, y_1) = (h, k)$ and $(x_2, y_2) \doteq (x, y)$.

$$\begin{aligned} \therefore\ r &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)} \\ &= \sqrt{(x-h)^2 + (y-k)^2} \qquad \text{or} \quad (x-h)^2 + (y-k)^2 = r^2 \end{aligned}$$

2. a. Textbook questions 8, 15, 16, 27, 29, and 32 of "Practice," p. 141

8. $(h, k) = (0, 0)$ and $r = 2\sqrt{2}$

$$\begin{aligned} \therefore\ x^2 + y^2 &= r^2 \\ x^2 + y^2 &= \left(2\sqrt{2}\right)^2 \quad \leftarrow \text{standard form of a circle with centre at the origin} \\ x^2 + y^2 &= 4(2) \\ x^2 + y^2 &= 8 \end{aligned}$$

15. $(h, k) = (5, -4)$ and $r = \sqrt{6}$

$$\begin{aligned} \therefore (x-h)^2 + (y-k)^2 &= r^2 \quad \leftarrow \text{standard form of a circle with centre } (h, k) \\ (x-5)^2 + [y-(-4)]^2 &= \left(\sqrt{6}\right)^2 \\ (x-5)^2 + (y+4)^2 &= 6 \end{aligned}$$

16. $(h, k) = (-6, 7)$ and $r = 3\sqrt{5}$

$$\begin{aligned} \therefore (x-h)^2 + (y-k)^2 &= r^2 \\ [x-(-6)]^2 + (y-7)^2 &= \left(3\sqrt{5}\right)^2 \\ (x+6)^2 + (y-7)^2 &= 9(5) \\ (x+6)^2 + (y-7)^2 &= 45 \end{aligned}$$

27. Find r^2 by substituting $(h, k) = (8, 2)$ and $(x, y) = (5, 0)$ into the standard form.

$$\begin{aligned}(x-h)^2+(y-k)^2&=r^2\\(5-8)^2+(0-2)^2&=r^2\\(-3)^2+(-2)^2&=r^2\\9+4&=r^2\\r^2&=13\end{aligned}$$

Therefore, the equation of the circle in standard form is $(x-8)^2+(y-2)^2=13$.

Write the equation in general form.

$$\begin{aligned}(x-8)^2+(y-2)^2&=13\\(x^2-16x+64)+(y^2-4y+4)&=13\\x^2+y^2-16x-4y+68&=13\\x^2+y^2-16x-4y+55&=0 \quad \leftarrow \text{general form}\end{aligned}$$

29. Find r^2 by substituting $(h, k) = (2, 3)$ and $(x, y) = (7, 2)$ into the standard form.

$$\begin{aligned}(x-h)^2+(y-k)^2&=r^2\\(7-2)^2+(2-3)^2&=r^2\\5^2+(-1)^2&=r^2\\25+1&=r^2\\r^2&=26\end{aligned}$$

Therefore, the equation of the circle in standard form is $(x-2)^2+(y-3)^2=26$.

Write the equation in general form.

$$\begin{aligned}(x-2)^2+(y-3)^2&=26\\(x^2-4x+4)+(y^2-6y+9)&=26\\x^2+y^2-4x-6y+13&=26\\x^2+y^2-4x-6y-13&=0 \quad \leftarrow \text{general form}\end{aligned}$$

Section 1: Activity 2 (continued)

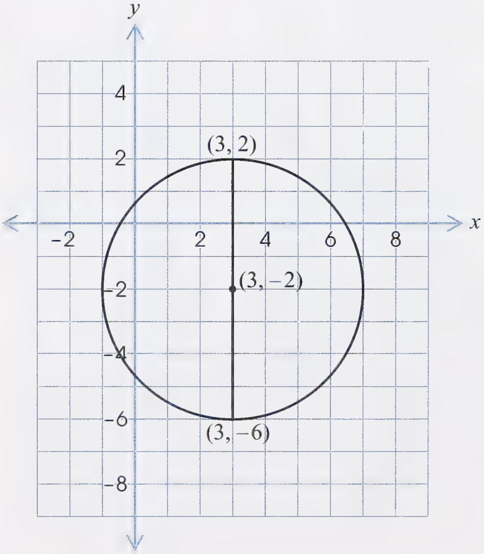
32. The midpoint of the diameter is the centre of the circle. Use the midpoint formula to determine the centre of the circle. Let $(x_1, y_1) = (3, -6)$ and $(x_2, y_2) \doteq (3, 2)$.

$$\begin{aligned}\therefore\ \text{Centre} &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\\ &= \left(\frac{3+3}{2}, \frac{-6+2}{2}\right)\\ &= \left(\frac{6}{2}, \frac{-4}{2}\right)\\ &= (3, -2)\end{aligned}$$

Find r^2 by substituting $(h, k) = (3, -2)$ and $(x, y) = (3, -6)$ into the standard form.

$$\begin{aligned}(x-h)^2+(y-k)^2 &= r^2\\ (3-3)^2+[-6-(-2)]^2 &= r^2\\ 0^2+(-4)^2 &= r^2\\ 0+16 &= r^2\\ r^2 &= 16\end{aligned}$$

Therefore, the equation of the circle in standard form is $(x-3)^2+(y+2)^2 = 16$.



b. Textbook questions 55 and 56 of "Applications and Problem Solving," p. 142

55. The circle $x^2 + y^2 = 25$ is translated 3 units to the left and 4 units up.

If (m, n) is a point on $x^2 + y^2 = 25$, then $(m - 3, n + 4)$ is a point on the translated circle.

If $(m - 3, n + 4) = (a, b)$, then

$$m - 3 = a \quad \text{and} \quad n + 4 = b$$
$$m = a + 3 \qquad\qquad n = b - 4$$

Therefore, $(m, n) = (a + 3, b - 4)$.

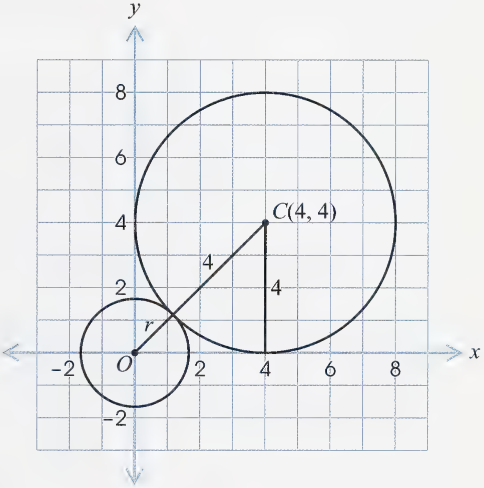
If (a, b) is a point on the translated circle, then $(a + 3, b - 4)$ is a point on the original circle.

56. Let r be the radius of the smaller circle.

$\therefore\ CO = r + 4$

Use the Pythagorean Theorem to find the radius of the smaller circle.

$$\begin{aligned}(CO)^2 &= 4^2 + 4^2\\ (r+4)^2 &= 16+16\\ (r+4)^2 &= 32\\ r+4 &= \sqrt{32}\\ r &= \sqrt{32}-4\\ &= 4\sqrt{2}-4\end{aligned}$$



Now, find the equation of the smaller circle.

$$\begin{aligned}x^2+y^2 &= r^2\\ x^2+y^2 &= \left(4\sqrt{2}-4\right)^2\\ x^2+y^2 &= 16(2)-16\sqrt{2}-16\sqrt{2}+16\\ x^2+y^2 &= 32-32\sqrt{2}+16\\ x^2+y^2 &= 48-32\sqrt{2}\end{aligned}$$

The equation of the smaller circle is $x^2 + y^2 = 48 - 32\sqrt{2}$.

Section 1: Activity 2 (continued)

3. Textbook questions 36, 37, and 39 of "Practice," p. 141

36.

$$
\begin{aligned}
x^2-6x+y^2-8y-39&=0\\
x^2-6x+y^2-8y&=39\\
\left[\left(x^2-6x+9\right)-9\right]+\left[\left(y^2-8y+16\right)-16\right]&=39 \quad \leftarrow \text{Complete the square.}\\
(x-3)^2-9+(y-4)^2-16&=39\\
(x-3)^2+(y-4)^2&=64
\end{aligned}
$$

$\therefore\ h=3,\ k=4,\ \text{and } r=\sqrt{64}=8$

The centre is at (3, 4), and the radius is 8.

37.

$$
\begin{aligned}
x^2-7x+y^2+7y&=17.75\\
\left[\left(x^2-7x+12.25\right)-12.25\right]+\left[\left(y^2+7y+12.25\right)-12.25\right]&=17.75 \quad \leftarrow \text{Complete the square.}\\
(x-3.5)^2-12.25+(y+3.5)^2-12.25&=17.75\\
(x-3.5)^2+(y+3.5)^2&=42.25
\end{aligned}
$$

$\therefore\ h=3.5,\ k=-3.5,\ \text{and } r=\sqrt{42.25}=6.5$

The centre is at (3.5, –3.5), and the radius is 6.5.

39.

$$
\begin{aligned}
x^2+8+y^2-8y&=0\\
x^2+y^2-8y&=-8\\
x^2+\left[\left(y^2-8y+16\right)-16\right]&=-8 \quad \leftarrow \text{Complete the square.}\\
x^2+(y-4)^2-16&=-8\\
x^2+(y-4)^2&=8
\end{aligned}
$$

$\therefore\ h=0,\ k=4,\ \text{and } r=\sqrt{8}=2\sqrt{2}$

The centre is at (0, 4), and the radius is $2\sqrt{2}$.

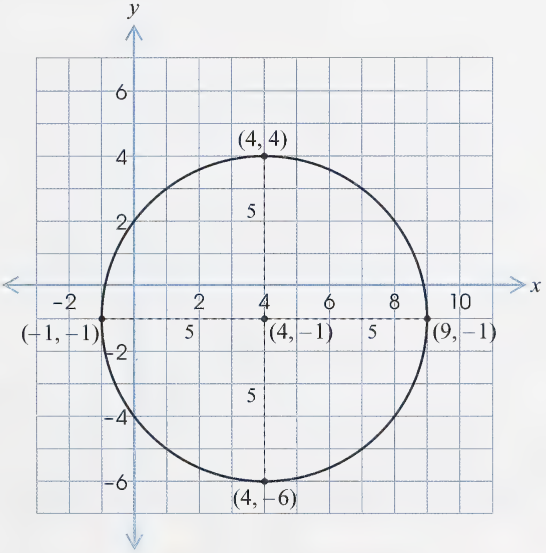
4. **a.** Because $A = C \neq 0$ and $B = 0$, the equation represents a circle or one of its degenerate forms.

Find the centre and radius of the circle.

$$\begin{aligned} x^2 + y^2 - 8x + 2y - 8 &= 0 \\ x^2 - 8x + y^2 + 2y &= 8 && \leftarrow \text{Group the terms.} \\ \left[(x^2 - 8x + 16) - 16\right] + \left[(y^2 + 2y + 1) - 1\right] &= 8 && \leftarrow \text{Complete the square.} \\ (x-4)^2 - 16 + (y+1)^2 - 1 &= 8 \\ (x-4)^2 + (y+1)^2 &= 25 \end{aligned}$$

The centre of the circle is (4, −1), and the radius is $\sqrt{25} = 5$.

Draw the circle, and identify the endpoints of the horizontal and vertical diameters.



The domain is $-1 \leq x \leq 9$, and the range is $-6 \leq y \leq 4$.

Section 1: Activity 2 (continued)

b.

$$x^2+y^2+12x+16y=0$$

$$x^2+12x+y^2+16y=0 \quad \leftarrow \text{Group the terms.}$$

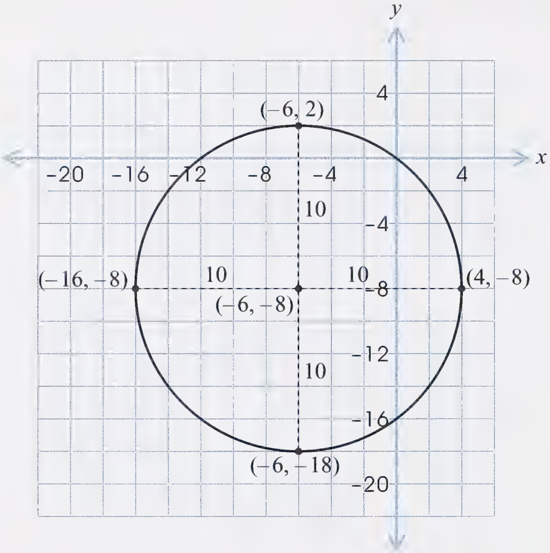
$$\left[\left(x^2+12x+36\right)-36\right]+\left[\left(y^2+16y+64\right)-64\right]=0 \quad \leftarrow \text{Complete the square.}$$

$$(x+6)^2-36+(y+8)^2-64=0$$

$$(x+6)^2+(y+8)^2=100$$

The centre is at (−6, −8), and the radius is $\sqrt{100}=10$.

Draw the circle, and identify the endpoints of the horizontal and vertical diameters.



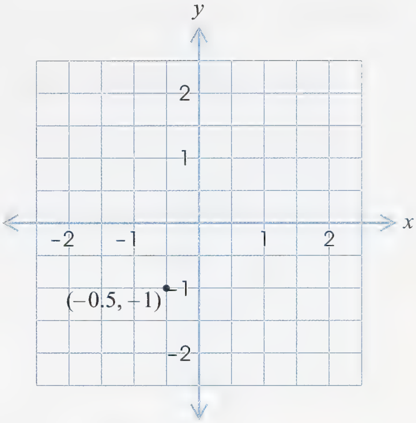
The domain is $-16 \le x \le 4$, and the range is $-18 \le y \le 2$.

c.

$$
\begin{aligned}
x^2+y^2+x+2y+1.25&=0\\
x^2+x+y^2+2y&=-1.25\\
\left[\left(x^2+x+0.25\right)-0.25\right]+\left[\left(y^2+2y+1\right)-1\right]&=-1.25\\
(x+0.5)^2-0.25+(y+1)^2-1&=-1.25\\
(x+0.5)^2+(y+1)^2&=0
\end{aligned}
$$

This is a point circle at (−0.5, −1).

The domain is $x=-0.5$, and the range is $y=-1$.



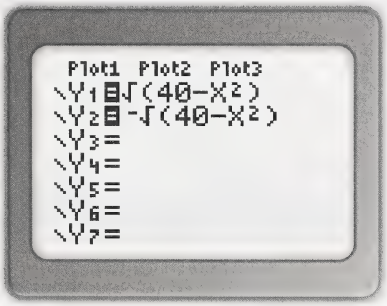
5. Textbook questions 40 and 45 of "Practice," p. 141

40.
$$
\begin{aligned}
x^2+y^2&=40\\
y^2&=40-x^2\\
y&=\pm\sqrt{40-x^2}
\end{aligned}
$$

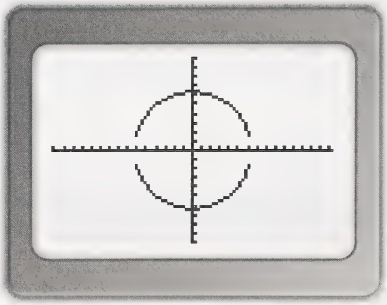
Use a square display to avoid distortion by pressing ZOOM 5.

Enter $y=\sqrt{40-x^2}$ as Y_1 and $y=-\sqrt{40-x^2}$ as Y_2.

Y= 2nd [√] 4 0 − X,T,Θ,n x^2) ENTER (−) 2nd [√]
4 0 − X,T,Θ,n x^2) GRAPH



```
 Plot1  Plot2  Plot3
\Y1■√(40-X²)
\Y2■-√(40-X²)
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



Section 1: Activity 2 (continued)

45. Convert the equation into standard form.

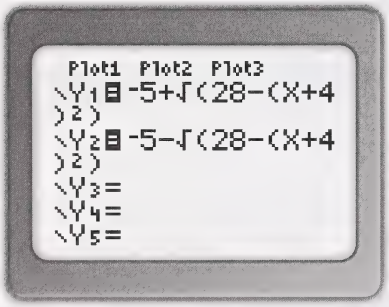
$$
\begin{aligned}
x^2+8x+y^2+10y+13&=0\\
x^2+8x+y^2+10y&=-13\\
\left[\left(x^2+8x+16\right)-16\right]+\left[\left(y^2+10y+25\right)-25\right]&=-13\\
(x+4)^2-16+(y+5)^2-25&=-13\\
(x+4)^2+(y+5)^2&=28
\end{aligned}
$$

Now, solve for y.

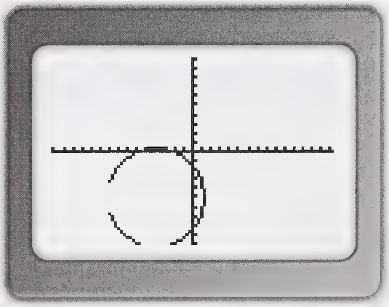
$$
\begin{aligned}
(y+5)^2&=28-(x+4)^2\\
y+5&=\pm\sqrt{28-(x+4)^2}\\
y&=-5\pm\sqrt{28-(x+4)^2}
\end{aligned}
$$

Enter $y=-5+\sqrt{28-(x+4)^2}$ as Y_1 and $y=-5-\sqrt{28-(x+4)^2}$ as Y_2.

Y= (–) 5 + 2nd [√] 2 8 − (X,T,Θ,n + 4) x^2) ENTER (–) 5 − 2nd [√] 2 8 − (X,T,Θ,n + 4) x^2) GRAPH



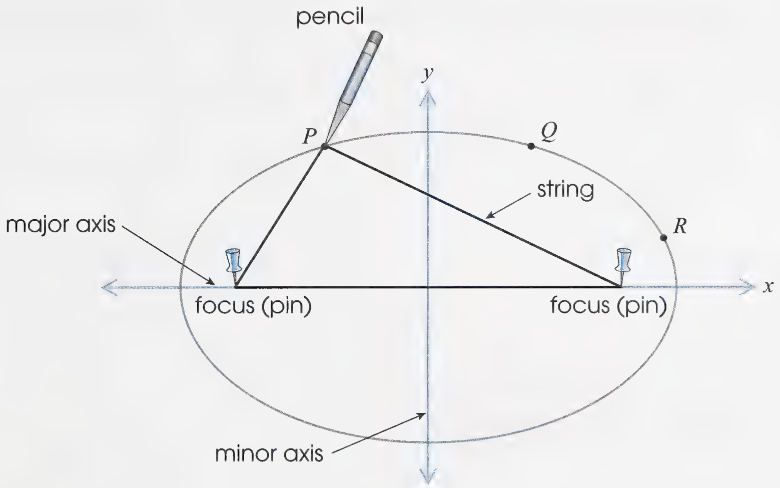
```
 Plot1 Plot2 Plot3
\Y1■-5+√(28-(X+4
)2)
\Y2■-5-√(28-(X+4
)2)
\Y3=
\Y4=
\Y5=
```



Section 1: Activity 3

1. a. **Textbook question "Explore: Use a Model," p. 143**

Your model should look somewhat similar to the following diagram.



b. **Textbook questions 1 to 7 of "Inquire," p. 143**

1. The major axis lies along the x-axis. If the foci were on the y-axis, the major axis would lie along the y-axis.

2. See diagram from "Explore: Use a Model."

3. Answers may vary. A sample answer is given.

   The sum of the two lengths is 2.6 cm + 5.0 cm = 7.6 cm.

4. Answers may vary. A sample answer is given.

   For point Q, the sum of the lengths from the two foci is 5.0 cm + 2.6 cm = 7.6 cm.

   For point R, the sum of the lengths from the two foci is 6.6 cm + 1.0 cm = 7.6 cm.

5. Regardless of the position of point, P, the sum will be the same.

6. Answers may vary. A sample answer is given.

   The major axis is about 7.6 cm in length, and the minor axis is about 4.8 cm in length.

7. The sum of the focal radii equals the length of the major axis.

Section 1: Activity 3 (continued)

2. Textbook questions 11 and 12 of "Practice," p. 150

11. Find the intercepts.

To find the x-intercepts, let $y = 0$.

$$\begin{aligned} \therefore\ \frac{x^2}{9}+\frac{y^2}{25}&=1 \\ \frac{x^2}{9}+\frac{0^2}{25}&=1 \\ x^2&=9 \\ x&=\pm 3 \end{aligned}$$

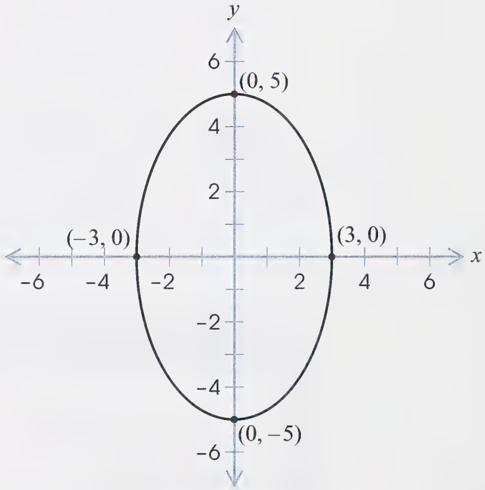
To find the y-intercepts, let $x = 0$.

$$\begin{aligned} \therefore\ \frac{x^2}{9}+\frac{y^2}{25}&=1 \\ \frac{0^2}{9}+\frac{y^2}{25}&=1 \\ y^2&=25 \\ y&=\pm 5 \end{aligned}$$

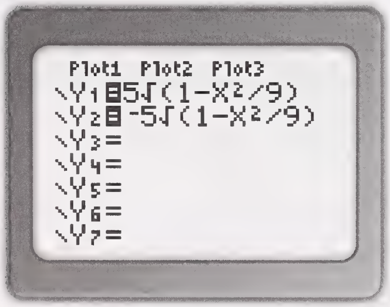
The x-intercepts are ± 3, and the y-intercepts are ± 5.

Sketch the ellipse from its intercepts. To graph this ellipse on your graphing calculator, solve the equation for y.

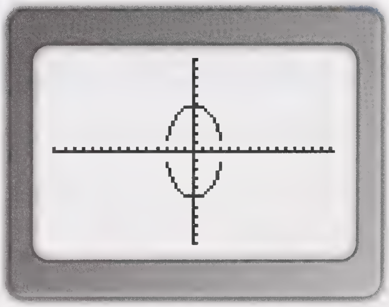
$$\begin{aligned} \frac{x^2}{9}+\frac{y^2}{25}&=1 \\ \frac{y^2}{25}&=1-\frac{x^2}{9} \\ y^2&=25\left(1-\frac{x^2}{9}\right) \\ y&=\pm 5\sqrt{1-\frac{x^2}{9}} \end{aligned}$$



Enter $y=5\sqrt{1-\frac{x^2}{9}}$ as Y_1 and $y=-5\sqrt{1-\frac{x^2}{9}}$ as Y_2.



```
Plot1 Plot2 Plot3
\Y1■5√(1-X²/9)
\Y2■-5√(1-X²/9)
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



12. Find the intercepts.

To find the x-intercepts, let $y = 0$.

$$\begin{aligned} \therefore\ \frac{x^2}{36}+y^2&=1 \\ \frac{x^2}{36}+0^2&=1 \\ x^2&=36 \\ x&=\pm 6 \end{aligned}$$

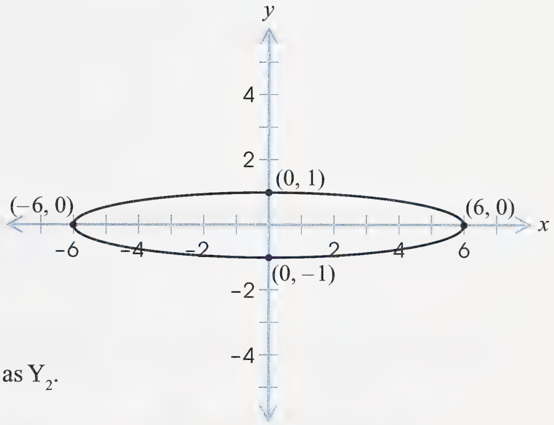
To find the y-intercepts, let $x = 0$.

$$\begin{aligned} \therefore\ \frac{x^2}{36}+y^2&=1 \\ \frac{0^2}{36}+y^2&=1 \\ y^2&=1 \\ y&=\pm 1 \end{aligned}$$

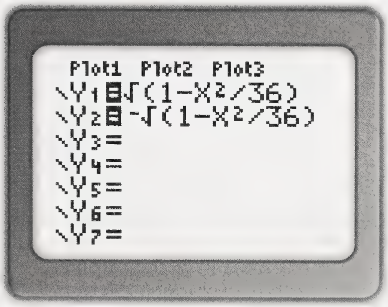
The x-intercepts are ±6, and the y-intercepts are ±1.

Sketch the ellipse from its intercepts. To graph this ellipse on your graphing calculator, solve the equation for y.

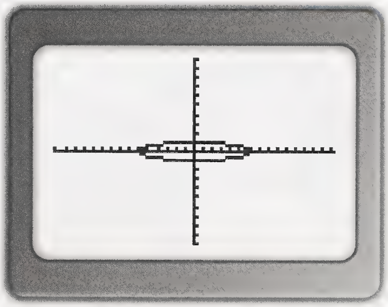
$$\begin{aligned} \frac{x^2}{36}+y^2&=1 \\ y^2&=1-\frac{x^2}{36} \\ y&=\pm\sqrt{1-\tfrac{x^2}{36}} \end{aligned}$$



Enter $y=\sqrt{1-\frac{x^2}{36}}$ as Y_1 and $y=-\sqrt{1-\frac{x^2}{36}}$ as Y_2.

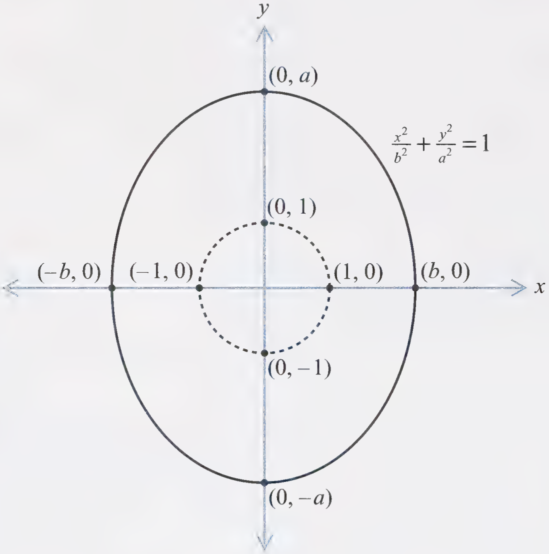


```
Plot1 Plot2 Plot3
\Y1=√(1-X²/36)
\Y2=-√(1-X²/36)
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



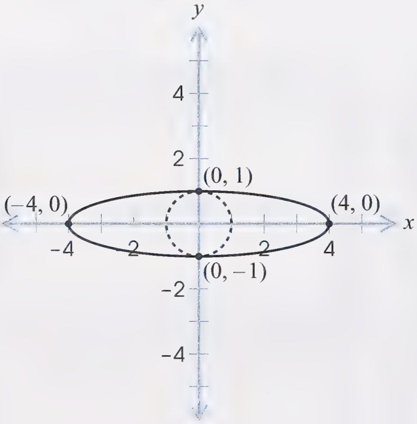
Section 1: Activity 3 (continued)

3. The graph of $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ can be obtained by stretching the graph of $x^2+y^2=1$ horizontally by a factor of b and vertically by a factor of a.



4. **Textbook question 32 of "Applications and Problem Solving," p. 151**

32. a.



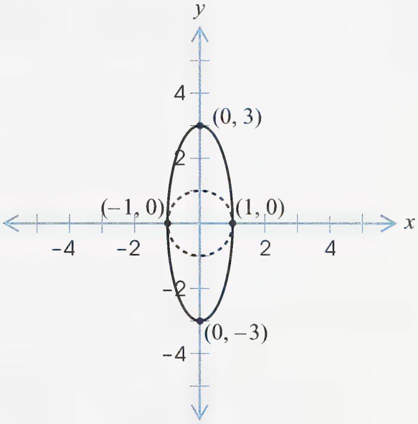
b. Refer to the answer to question 32.a.

c. Since $\frac{x^2}{16}+y^2=1$ is the same as $\left(\frac{x}{4}\right)^2+y^2=1$, the graph of $\frac{x^2}{16}+y^2=1$ is the graph of $x^2+y^2=1$ stretched horizontally by a factor of 4.

d. The mapping of $x^2+y^2=1$ to $\frac{x^2}{16}+y^2=1$ is $(x, y) \rightarrow (\mathbf{4}x, \mathbf{1}y)$.

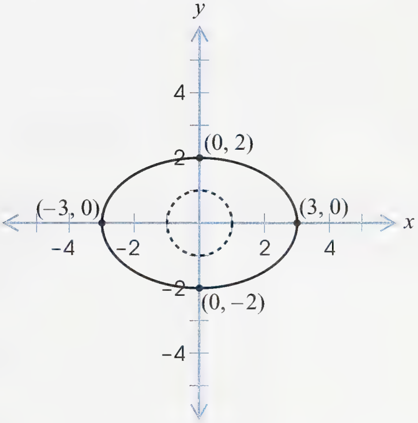
e. Since $x^2+\frac{y^2}{9}=1$ is the same as $x^2+\left(\frac{y}{3}\right)^2=1$, the graph of $x^2+\frac{y^2}{9}=1$ is the graph of $x^2+y^2=1$ stretched vertically by a factor of 3.

The mapping of $x^2+y^2=1$ to $x^2+\frac{y^2}{9}=1$ is $(x, y) \rightarrow (\mathbf{1}x, \mathbf{3}y)$.



f. Since $\frac{x^2}{9}+\frac{y^2}{4}=1$ is the same as $\left(\frac{x}{3}\right)^2+\left(\frac{y}{2}\right)^2=1$, the graph of $\frac{x^2}{9}+\frac{y^2}{4}=1$ is the graph of $x^2+y^2=1$ stretched horizontally by a factor of 3 and vertically by a factor of 2.

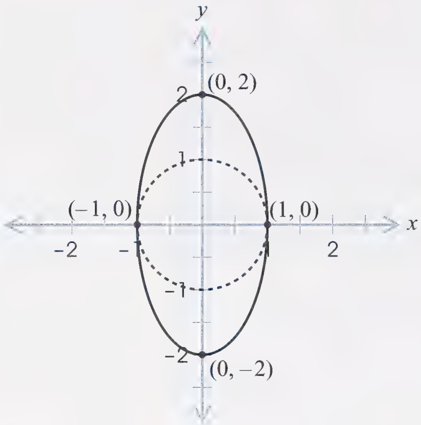
The mapping of $x^2+y^2=1$ to $\frac{x^2}{9}+\frac{y^2}{4}=1$ is $(x, y) \rightarrow (\mathbf{3}x, \mathbf{2}y)$.



g. The mapping of $x^2+y^2=1$ to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $(x, y) \rightarrow (ax, by)$.

Section 1: Activity 3 (continued)

5. **a.** Sketch the ellipse and the unit circle.



Because the ellipse is the unit circle stretched vertically by a factor of 2, substitute $\frac{y}{2}$ for y into the standard form of the unit circle.

$$x^2 + y^2 = 1$$

$$x^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$x^2 + \frac{y^2}{4} = 1 \quad \leftarrow \text{standard form}$$

Express the equation in general form.

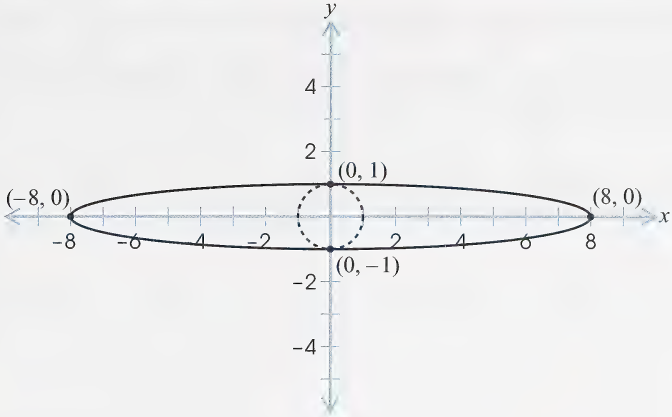
$$x^2 + \frac{y^2}{4} = 1$$

$$4\left(x^2 + \frac{y^2}{4}\right) = 4(1) \quad \leftarrow \text{Multiply both sides by the LCD, 4.}$$

$$4x^2 + y^2 = 4$$

$$4x^2 + y^2 - 4 = 0 \quad \leftarrow \text{general form}$$

b. Sketch the ellipse and the unit circle.



Because the ellipse is the unit circle stretched horizontally by a factor of 8, substitute $\frac{x}{8}$ for x into the standard form of the unit circle.

$$x^2 + y^2 = 1$$

$$\left(\frac{x}{8}\right)^2 + y^2 = 1$$

$$\frac{x^2}{64} + y^2 = 1 \quad \leftarrow \text{standard form}$$

Express the equation in general form.

$$\frac{x^2}{64} + y^2 = 1$$

$$64\left(\frac{x^2}{64} + y^2\right) = 64(1) \quad \leftarrow \text{Multiply both sides by the LCD, 64.}$$

$$x^2 + 64y^2 = 64$$

$$x^2 + 64y^2 - 64 = 0 \quad \leftarrow \text{general form}$$

Section 1: Activity 3 (continued)

6. Textbook questions 33, 34, and 36 of "Applications and Problem Solving," p. 151

33. The graph of this ellipse is the graph of the unit circle stretched horizontally by a factor of 2 and vertically by a factor of 5.

To stretch the unit circle horizontally by a factor of 2 and vertically by a factor of 5, substitute $\frac{x}{2}$ for x and $\frac{y}{5}$ for y into the equation of the circle.

$$x^2 + y^2 = 1$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$$

$$\frac{x^2}{4} + \frac{y^2}{25} = 1 \quad \leftarrow \text{standard form}$$

34. The major axis of this ellipse is horizontal and 4 units in length. Because the diameter of the unit circle is only 2 units long, the circle has been stretched horizontally by a factor of 2.

The minor axis of the ellipse is vertical and 2 units in length, the same length as the diameter of the unit circle. The ellipse has not been stretched vertically.

The centre of the ellipse is (5, 7). Therefore, it has been translated 5 units to the right and 7 units up.

To stretch the unit circle horizontally by a factor of 2, substitute $\frac{x}{2}$ for x into the equation of the circle.

$$x^2 + y^2 = 1$$

$$\left(\frac{x}{2}\right)^2 + y^2 = 1$$

$$\frac{x^2}{4} + y^2 = 1$$

To translate this ellipse 5 units to the right and 7 units up, substitute $(x - 5)$ for x and $(y - 7)$ for y.

$$\frac{x^2}{4} + y^2 = 1$$

$$\frac{(x-5)^2}{4} + (y-7)^2 = 1 \quad \leftarrow \text{standard form}$$

36. The horizontal axis, the minor axis in this case, of this ellipse is 4 units in length. Because the diameter of the unit circle is only 2 units long, the circle has been stretched horizontally by a factor of 2.

The vertical axis, the major axis in this case, of this ellipse is 6 units in length. Because the diameter of the unit circle is only 2 units long, the circle has been stretched vertically by a factor of 3.

The centre of the ellipse is (–3, 6). Therefore, it has been translated 3 units to the left and 6 units up.

To stretch the unit circle horizontally by a factor of 2 and vertically by a factor of 3, substitute $\frac{x}{2}$ for x and $\frac{y}{3}$ for y into the equation of the circle.

$$\begin{aligned} x^2 + y^2 &= 1 \\ \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 &= 1 \\ \frac{x^2}{4} + \frac{y^2}{9} &= 1 \end{aligned}$$

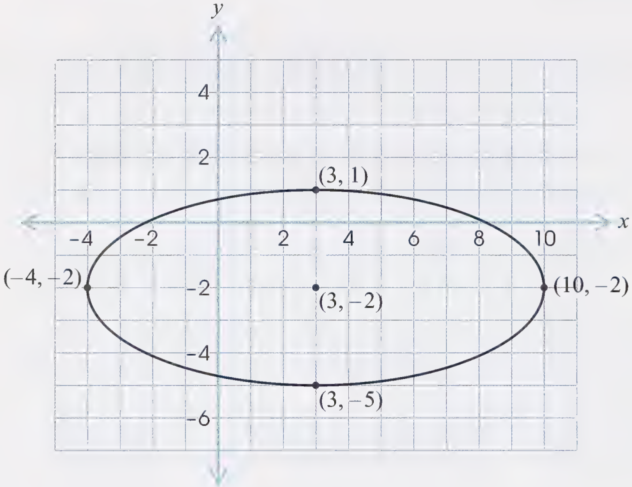
To translate this ellipse 3 units to the left and 6 units up, substitute $(x + 3)$ for x and $(y - 6)$ for y.

$$\begin{aligned} \frac{x^2}{4} + \frac{y^2}{9} &= 1 \\ \frac{(x+3)^2}{4} + \frac{(y-6)^2}{9} &= 1 \quad \leftarrow \text{standard form} \end{aligned}$$

Section 1: Activity 3 (continued)

7. **a. Textbook question 21 of "Practice," p. 150**

21. Plot the points and draw the ellipse.



The centre is at (3, –2). So, $h = 3$ and $k = -2$.

The major axis is parallel to the x-axis and 14 units in length. So, $a = 7$.

The minor axis is parallel to the y-axis and 6 units in length. So, $b = 3$.

Substitute $h = 3$, $k = -2$, $a = 7$, and $b = 3$ into the standard form of an ellipse.

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-3)^2}{7^2}+\frac{[y-(-2)]^2}{3^2}=1$$

$$\frac{(x-3)^2}{49}+\frac{(y+2)^2}{9}=1 \quad \leftarrow \text{standard form}$$

$$9(x-3)^2+49(y+2)^2=49\times 9$$

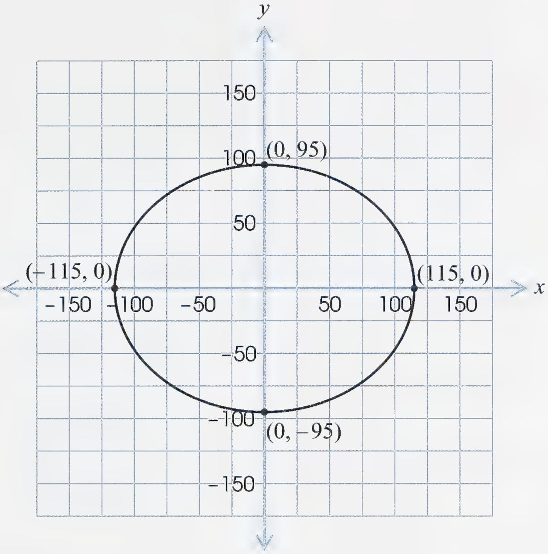
$$9(x^2-6x+9)+49(y^2+4y+4)=441$$

$$9x^2-54x+81+49y^2+196y+196=441$$

$$9x^2+49y^2-54x+196y-164=0 \quad \leftarrow \text{general form}$$

b. Textbook question 40.a. of "Applications and Problem Solving," p. 152

40. a. Centre the ellipse at the origin and position the major axis along the x-axis.



The major axis is along the x-axis and 230 m long. So, $a = 115$.

The minor axis is along the y-axis and 190 m long. So, $b = 95$.

Substitute $a = 115$ and $b = 95$ into the standard form of an ellipse centred at the origin.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{115^2} + \frac{y^2}{95^2} = 1$$

$$\frac{x^2}{13\,225} + \frac{y^2}{9025} = 1$$

The equation of the ellipse formed by the base of the roof is $\frac{x^2}{13\,225} + \frac{y^2}{9025} = 1$.

Section 1: Activity 3 (continued)

8. a.

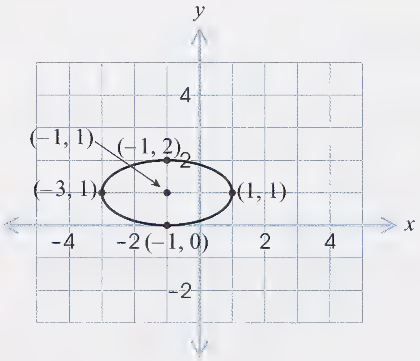
$$
\begin{aligned}
x^2+4y^2+2x-8y+1&=0\\
x^2+2x+4y^2-8y&=-1 && \leftarrow \text{Group the terms.}\\
\left(x^2+2x\right)+4\left(y^2-2y\right)&=-1 && \leftarrow \text{Remove the coefficient from } y^2.\\
\left[\left(x^2+2x+1\right)-1\right]+4\left[\left(y^2-2y+1\right)-1\right]&=-1 && \leftarrow \text{Complete the square.}\\
\left[(x+1)^2-1\right]+4\left[(y-1)^2-1\right]&=-1\\
(x+1)^2-1+4(y-1)^2-4&=-1\\
(x+1)^2+4(y-1)^2&=4\\
\frac{(x+1)^2}{4}+\frac{\cancel{4}(y-1)^2}{\cancel{4}}&=\frac{4}{4} && \leftarrow \text{Divide each term by 4.}\\
\frac{(x+1)^2}{4}+\frac{(y-1)^2}{1}&=1
\end{aligned}
$$

Compare this equation with $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

$\therefore\ h=-1,\ k=1,\ a=\sqrt{4}=2,\ \text{and } b=\sqrt{1}=1$

The centre of this ellipse is at $(-1, 1)$; the major axis is parallel to the x-axis, and its endpoints are $a = 2$ units from the centre; and the minor axis is parallel to the y-axis, and its endpoints are $b = 1$ unit from the centre.

Now, draw the ellipse.



b.

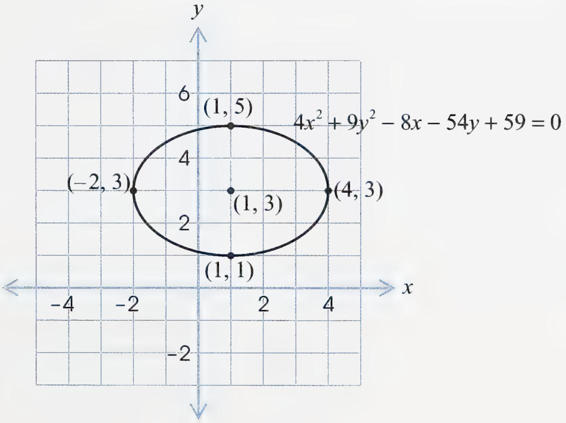
$$
\begin{aligned}
4x^2+9y^2-8x-54y+49&=0\\
4x^2-8x+9y^2-54y&=-49\\
4\left(x^2-2x\right)+9\left(y^2-6y\right)&=-49\\
4\left[\left(x^2-2x+1\right)-1\right]+9\left[\left(y^2-6y+9\right)-9\right]&=-49\\
4\left[(x-1)^2-1\right]+9\left[(y-3)^2-9\right]&=-49\\
4(x-1)^2-4+9(y-3)^2-81&=-49\\
4(x-1)^2+9(y-3)^2&=36\\
\frac{\overset{1}{\cancel{4}}(x-1)^2}{\underset{9}{\cancel{36}}}+\frac{\overset{1}{\cancel{9}}(y-3)^2}{\underset{4}{\cancel{36}}}&=\frac{36}{36}\\
\frac{(x-1)^2}{9}+\frac{(y-3)^2}{4}&=1
\end{aligned}
$$

Compare this equation with $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

$\therefore\ h=1,\ k=3,\ a=\sqrt{9}=3,$ and $b=\sqrt{4}=2$

The centre of this ellipse is at (1, 3); the major axis is parallel to the x-axis, and its endpoints are $a = 3$ units from the centre; and the minor axis is parallel to the y-axis, and its endpoints are $b = 2$ units from the centre.

Now, draw the ellipse.



Section 1: Activity 3 (continued)

c.

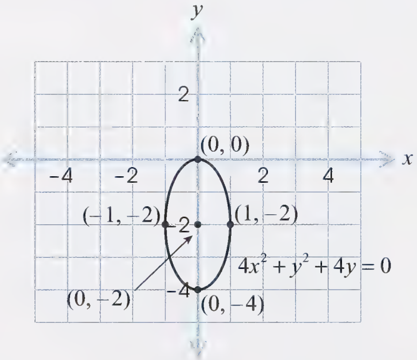
$$
\begin{aligned}
4x^2 + y^2 + 4y &= 0\\
4x^2 + \left[\left(y^2 + 4y + 4\right) - 4\right] &= 0\\
4x^2 + (y+2)^2 - 4 &= 0\\
4x^2 + (y+2)^2 &= 4\\
\frac{\cancel{4}x^2}{\cancel{4}} + \frac{(y+2)^2}{4} &= \frac{4}{4}\\
\frac{x^2}{1} + \frac{(y+2)^2}{4} &= 1
\end{aligned}
$$

Compare this equation with $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$.

$\therefore\ h = 0,\ k = -2,\ a = \sqrt{4} = 2,$ and $b = \sqrt{1} = 1$

The centre of this ellipse is at (0, –2); the major axis lies along the y-axis, and its endpoints are $a = 2$ units from the centre; and the minor axis is parallel to the x-axis, and its endpoints are $b = 1$ unit from the centre.

Now, draw the ellipse.



9. To stretch the graph vertically by a factor of 3 about the line $y = -4$, follow these steps:

Step 1: Move the graph up 4 units so the graph is stretched about the x-axis.

$$\frac{(x-1)^2}{5^2}+\frac{(y+3)^2}{2^2}=1$$

$$\frac{(x-1)^2}{5^2}+\frac{[(y-4)+3]^2}{2^2}=1$$

$$\frac{(x-1)^2}{5^2}+\frac{(y-1)^2}{2^2}=1$$

Step 2: Stretch the graph vertically by a factor of 3 about the x-axis. Divide y by 3.

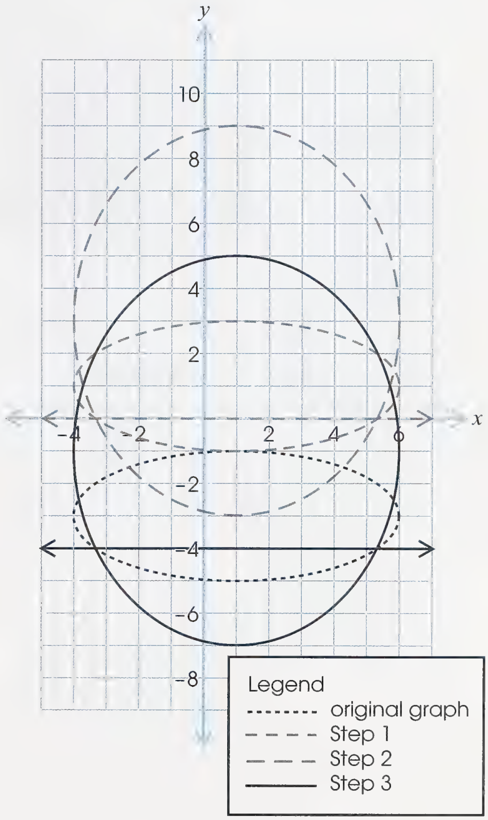
$$\frac{(x-1)^2}{5^2}+\frac{\left(\frac{y}{3}-1\right)^2}{2^2}=1$$

$$\frac{(x-1)^2}{5^2}+\frac{\left(\frac{y-3}{3}\right)^2}{2^2}=1$$

$$\frac{(x-1)^2}{5^2}+\frac{\frac{(y-3)^2}{9}}{4}=1$$

$$\frac{(x-1)^2}{5^2}+\frac{(y-3)^2}{36}=1$$

$$\frac{(x-1)^2}{5^2}+\frac{(y-3)^2}{6^2}=1$$



Step 3: Move the graph down 4 units so it is in its original place.

$$\frac{(x-1)^2}{5^2}+\frac{(y-3)^2}{6^2}=1$$

$$\frac{(x-1)^2}{5^2}+\frac{[(y+4)-3]^2}{6^2}=1$$

$$\frac{(x-1)^2}{5^2}+\frac{(y+1)^2}{6^2}=1$$

The equation of the resulting graph is

$$\frac{(x-1)^2}{5^2}+\frac{(y+1)^2}{6^2}=1.$$

Section 1: Activity 3 (continued)

10. Textbook question "NUMBER POWER," p. 152

$$5 - 6 + 7 = 6$$
$$9 - 2 - 1 = 6$$
$$8 \div 4 \times 3 = 6$$

Section 1: Follow-up Activities

Extra Help

1. Compare $\frac{(x-1)^2}{4}+\frac{(y+2)^2}{9}=1$ with $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$.

$\therefore\ h=1,\ k=-2,\ a=\sqrt{9}=3,$ and $b=\sqrt{4}=2$

The centre, (h, k), of the ellipse is at $(1, -2)$.

Because $b = 2$, the endpoints of the minor axis are located at $(1-2, -2) = (-1, -2)$ and $(1+2, -2) = (3, -2)$. Therefore, the domain is $-1 \le x \le 3$.

Because $a = 3$, the endpoints of the major axis are located at $(1, -2-3) = (1, -5)$ and $(1, -2+3) = (1, 1)$. Therefore, the range is $-5 \le y \le 1$.

Convert the equation into general form.

$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{9}=1$$

$$\overset{9}{\cancel{36}}\left[\frac{(x-1)^2}{\underset{1}{\cancel{4}}}\right]+\overset{4}{\cancel{36}}\left[\frac{(y+2)^2}{\underset{1}{\cancel{9}}}\right]=36(1) \quad \leftarrow \text{Multiply each term by the LCD, 36.}$$

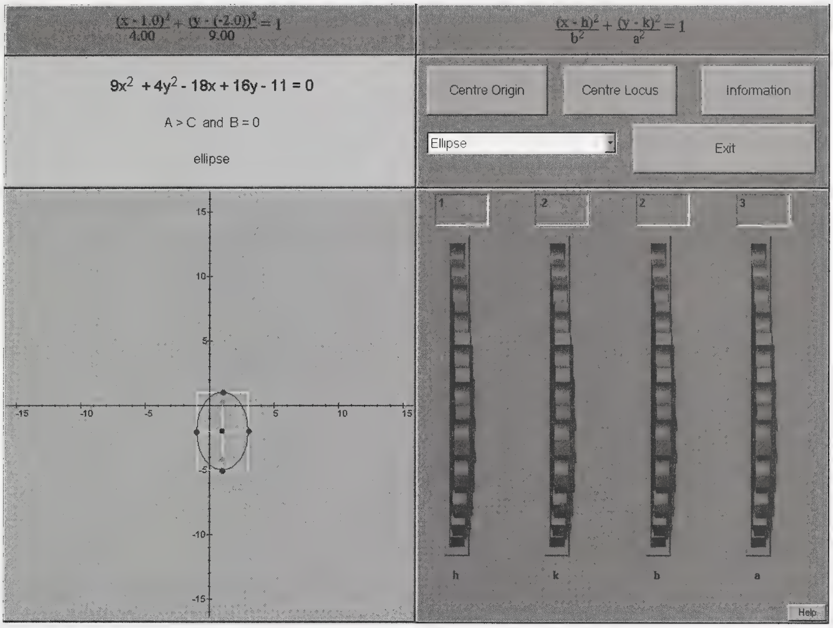
$$9(x-1)^2+4(y+2)^2=36$$

$$9(x^2-2x+1)+4(y^2+4y+4)=36$$

$$9x^2-18x+9+4y^2+16y+16=36$$

$$9x^2+4y^2-18x+16y-11=0 \quad \leftarrow \text{general form}$$

Check



$\frac{(x - 1.0)^2}{4.00} + \frac{(y - (-2.0))^2}{9.00} = 1$

$9x^2 + 4y^2 - 18x + 16y - 11 = 0$

A > C and B = 0

ellipse

$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$

Ellipse

2.

$$
\begin{aligned}
x^2 + y^2 + 2x - 4y - 20 &= 0\\
x^2 + 2x + y^2 - 4y &= 20\\
(x^2 + 2x) + (y^2 - 4y) &= 20\\
(x^2 + 2x + 1) - 1 + (y^2 - 4y + 4) - 4 &= 20\\
(x+1)^2 - 1 + (y-2)^2 - 4 &= 20\\
(x+1)^2 + (y-2)^2 &= 25 \quad \leftarrow \text{standard form}
\end{aligned}
$$

$\therefore\ h = -1,\ k = 2,$ and $r = \sqrt{25} = 5$

The centre, (h, k), of the circle is at $(-1, 2)$.

Because $r = 5$, the domain is

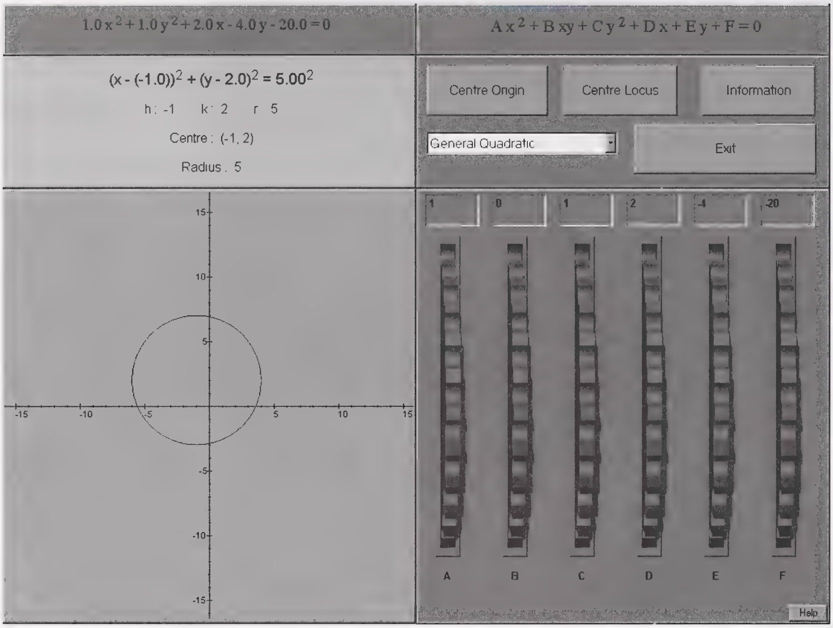
$$
\begin{aligned}
-1 - 5 \le x &\le -1 + 5\\
-6 \le x &\le 4
\end{aligned}
$$

Extra Help (continued)

Because $r = 5$, the range is

$$\begin{aligned} 2 - 5 \leq y &\leq 2 + 5 \\ -3 \leq y &\leq 7 \end{aligned}$$

Check



$1.0x^2 + 1.0y^2 + 2.0x - 4.0y - 20.0 = 0$

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

$(x - (-1.0))^2 + (y - 2.0)^2 = 5.00^2$

h: -1 k: 2 r 5

Centre: (-1, 2)

Radius . 5

General Quadratic

A: 1 B: 0 C: 1 D: 2 E: -4 F: -20

3. The centre is at (2, 3); so $h = 2$ and $k = 3$.

The major axis is vertical and 6 units in length; so, $a = 3$.

The minor axis is horizontal and 4 units in length; so, $b = 2$.

Substitute $h = 2$, $k = 3$, $a = 3$, and $b = 2$ into the standard form of an ellipse.

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{(x-2)^2}{2^2} + \frac{(y-3)^2}{3^2} = 1$$

$$\frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} = 1 \qquad \leftarrow \text{standard form}$$

$$\overset{9}{\cancel{36}}\left[\frac{(x-2)^2}{\underset{1}{\cancel{4}}}\right] + \overset{4}{\cancel{36}}\left[\frac{(y-3)^2}{\underset{1}{\cancel{9}}}\right] = 36(1) \qquad \leftarrow \text{Multiply each term by the LCD, 36.}$$

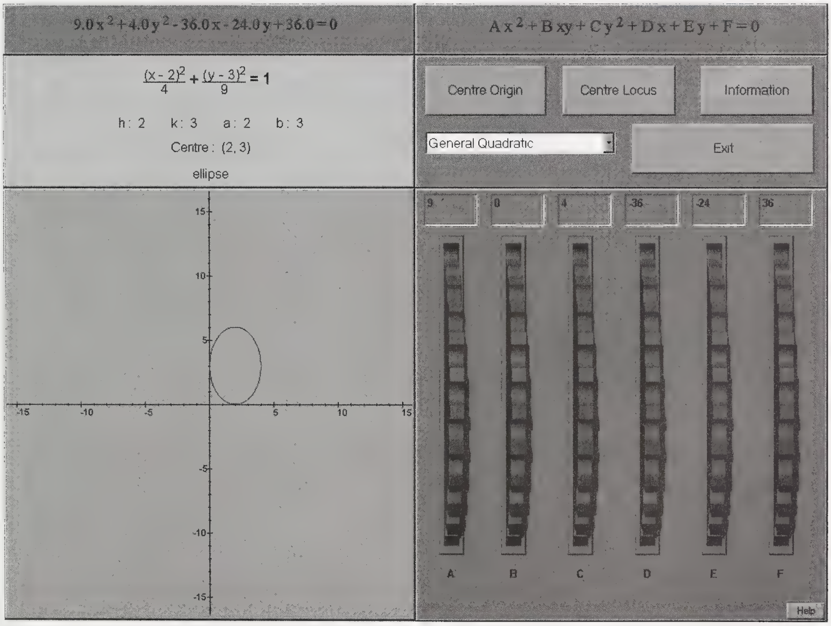
$$9(x-2)^2 + 4(y-3)^2 = 36$$

$$9(x^2 - 4x + 4) + 4(y^2 - 6y + 9) = 36$$

$$9x^2 - 36x + 36 + 4y^2 - 24y + 36 = 36$$

$$9x^2 + 4y^2 - 36x - 24y + 36 = 0 \qquad \leftarrow \text{general form}$$

Check



$9.0x^2 + 4.0y^2 - 36.0x - 24.0y + 36.0 = 0$

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

$\frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} = 1$

h: 2 k: 3 a: 2 b: 3

Centre: (2, 3)

ellipse

General Quadratic

A: 9, B: 0, C: 4, D: -36, E: -24, F: 36

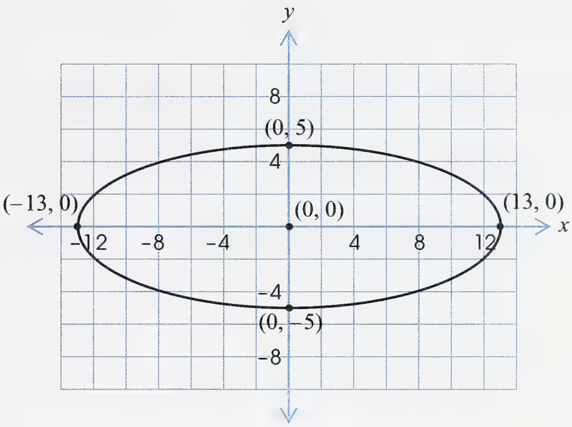
Enrichment

1. Let (0, 3) be F_1 and (0, –3) be F_2.

$$
\begin{aligned}
\therefore\ PF_1 + PF_2 &= 10\\
\sqrt{(x-0)^2+(y-3)^2}+\sqrt{(x-0)^2+\left[y-(-3)\right]^2} &= 10\\
\sqrt{x^2+(y-3)^2}+\sqrt{x^2+(y+3)^2} &= 10\\
\sqrt{x^2+(y-3)^2} &= 10-\sqrt{x^2+(y+3)^2}\\
\left(\sqrt{x^2+(y-3)^2}\right)^2 &= \left(10-\sqrt{x^2+(y+3)^2}\right)^2\\
\cancel{x}^2+(y-3)^2 &= 100-20\sqrt{x^2+(y+3)^2}+\cancel{x}^2+(y+3)^2\\
(y-3)^2 &= 100-20\sqrt{x^2+(y+3)^2}+(y+3)^2\\
\cancel{y}^2-6y+\cancel{9} &= 100-20\sqrt{x^2+y^2+6y+9}+\cancel{y}^2+6y+\cancel{9}\\
-6y &= 100-20\sqrt{x^2+y^2+6y+9}+6y\\
20\sqrt{x^2+y^2+6y+9} &= 100+12y\\
5\sqrt{x^2+y^2+6y+9} &= 25+3y \quad \leftarrow \text{Divide each term by the LCM, 4.}\\
\left(5\sqrt{x^2+y^2+6y+9}\right)^2 &= (25+3y)^2\\
25\left(x^2+y^2+6y+9\right) &= 625+150y+9y^2\\
25x^2+25y^2+\cancel{150y}+225 &= 625+\cancel{150y}+9y^2\\
25x^2+25y^2+225 &= 625+9y^2\\
25x^2+16y^2 &= 400\\
\frac{\overset{1}{\cancel{25}}\,x^2}{\underset{16}{\cancel{400}}}+\frac{\overset{1}{\cancel{16}}\,y^2}{\underset{25}{\cancel{400}}} &= \frac{400}{400}\\
\frac{x^2}{16}+\frac{y^2}{25} &= 1
\end{aligned}
$$

2. Draw the ellipse by comparing

$\frac{x^2}{169}+\frac{y^2}{25}$ with $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$



$\therefore\ h=0,\ k=0,\ a=\sqrt{169}=13,$ and $b=\sqrt{25}=5$

To determine the location of the foci, use

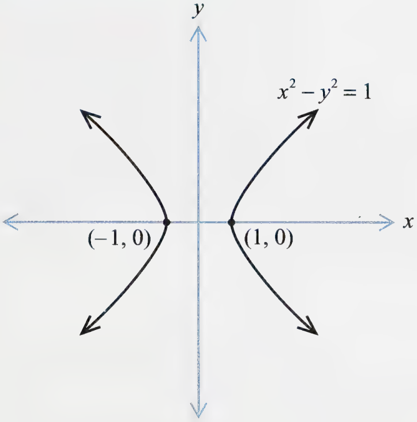
$$\begin{aligned} a^2 &= b^2+c^2 \\ 169 &= 25+c^2 \\ 144 &= c^2 \\ c^2 &= 144 \\ c &= 12 \end{aligned}$$

Recall that a, b, and c are non-negative.

Therefore, the foci are located on the major axis at $(\pm c, 0) = (\pm 12, 0)$.

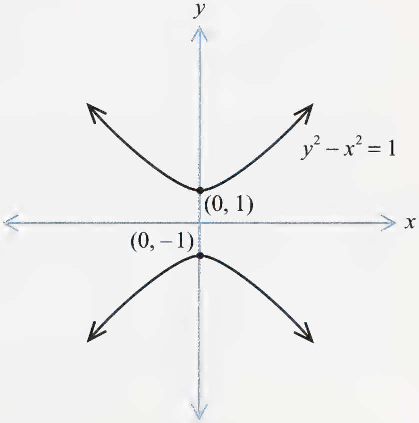
Section 2: Activity 1

1. **a.**



The domain is $x \le -1$ or $x \ge 1$, and the range is the set of reals.

b.



The domain is the set of reals, and the range is $y \le -1$ or $y \ge 1$.

Section 2: Activity 1 (continued)

2. **a.** The equation $\frac{x^2}{25} - y^2 = 1$ is similar in form to $x^2 - y^2 = 1$.

Because $\frac{x^2}{25} - y^2 = 1$ is the same as $\left(\frac{x}{5}\right)^2 - y^2 = 1$, the graph of $x^2 - y^2 = 1$ has been stretched horizontally by a factor of 5.

Because the hyperbola opens to the left and to the right, use the horizontal stretch factor of 5 to locate the vertices.

$$(-1 \times 5, 0) = (-5, 0) \text{ and } (1 \times 5, 0) = (5, 0)$$

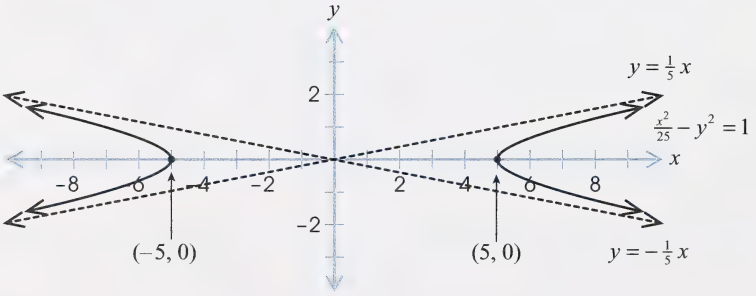
The slopes of the asymptotes of $x^2 - y^2 = 1$ are $\pm\frac{1}{1}$. Therefore, the slopes of the new asymptotes are

$$\begin{aligned} m &= \pm\frac{1}{1 \times \text{horizontal stretch}} \\ &= \pm\frac{1}{5} \end{aligned}$$

Because the asymptotes pass through the origin, the equations are

$$\begin{aligned} y &= mx \\ &= \pm\frac{1}{5}x \end{aligned}$$

Sketch the curve by first locating the vertices. Then draw the asymptotes from their slopes. Finally, sketch the branches of the hyperbola opening to the left and to the right from the vertices and between the asymptotes.



b. The equation $\frac{y^2}{4}-\frac{x^2}{25}=1$ is similar to $y^2-x^2=1$.

Because $\frac{y^2}{4}-\frac{x^2}{25}=1$ is the same as $\left(\frac{y}{2}\right)^2-\left(\frac{x}{5}\right)^2=1$, the graph of $y^2-x^2=1$ is stretched horizontally by a factor of 5 and vertically by a factor of 2.

Because the hyperbola opens upward and downward, use the vertical stretch factor of 2 to locate the vertices.

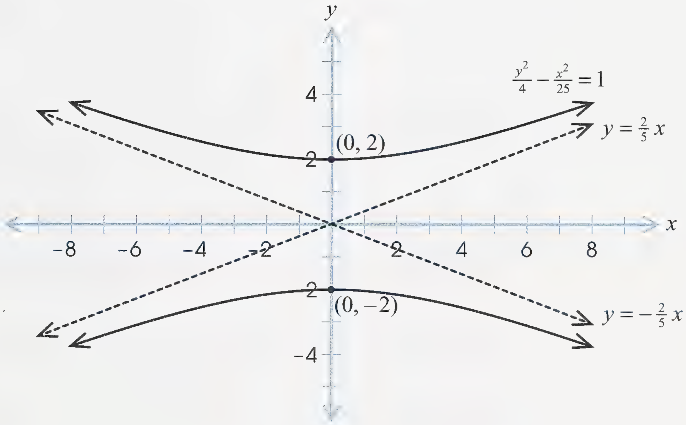
$$(0, -1\times 2)=(0,-2) \text{ and } (0, 1\times 2)=(0, 2)$$

The slopes of the asymptotes of $y^2-x^2=1$ are $\pm\frac{1}{1}$. Therefore, the slopes of the new asymptotes are

$$\begin{aligned} m &= \pm\frac{1\times\text{vertical stretch}}{1\times\text{horizontal stretch}} \\ &= \pm\frac{2}{5} \end{aligned}$$

Thus, the asymptotes are $y=\pm\frac{2}{5}x$.

Sketch the curve by first locating the vertices. Then draw the asymptotes from their slopes. Finally, sketch the branches of the hyperbola opening upward and downward from the vertices and between the asymptotes.



3. a. Textbook questions 7, 9, 13, and 31 of "Practice," pp. 160 and 161

7. Compare $\frac{(x-1)^2}{144}-\frac{y^2}{64}=1$ with $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.

$\therefore\ h=1,\ k=0,\ a=\sqrt{144}=12,$ and $b=\sqrt{64}=8$

a. The centre of the hyperbola is at (1, 0).

Section 2: Activity 1 (continued)

b. The transverse axis is parallel to the x-axis, and its length is $2(12) = 24$ units. The conjugate axis is parallel to the y-axis, and its length is $2(8) = 16$ units.

c. The vertices are located at $(1 - 12, 0) = (-11, 0)$ and $(1 + 12, 0) = (13, 0)$.

d.
$$\begin{aligned}\pm\frac{b}{a} &= \pm\frac{8}{12}\\ &= \pm\frac{2}{3}\end{aligned}$$

The slopes of the asymptotes are $\pm\frac{2}{3}$.

9. Compare $\frac{(y+4)^2}{49} - \frac{x^2}{100} = 1$ with $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.

$\therefore\ h = 0,\ k = -4,\ a = \sqrt{49} = 7,$ and $b = \sqrt{100} = 10$

a. The centre of the hyperbola is at $(0, -4)$.

b. The transverse axis is parallel to the y-axis, and its length is $2(7) = 14$ units. The conjugate axis is parallel to the x-axis, and its length is $2(10) = 20$ units.

c. The vertices are located $(0, -4 - 7) = (0, -11)$ and $(0, -4 + 7) = (0, 3)$.

d. The slopes of the asymptotes are $\pm\frac{a}{b} = \pm\frac{7}{10}$.

13. Compare $\frac{(y+6)^2}{9} - \frac{(x+2)^2}{256} = 1$ with $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.

$\therefore\ h = -2,\ k = -6,\ a = \sqrt{9} = 3,$ and $b = \sqrt{256} = 16$

a. The centre of the hyperbola is at $(-2, -6)$.

b. The transverse axis is parallel to the y-axis, and its length is $2(3) = 6$ units. The conjugate axis is parallel to the x-axis, and its length is $2(16) = 32$ units.

c. The vertices are located at $(-2, -6 - 3) = (-2, -9)$ and $(-2, -6 + 3) = (-2, -3)$.

d. The slopes of the asymptotes are $\pm\frac{a}{b}$ or $\pm\frac{3}{16}$.

31. $4y^2-25(x-2)^2=100$

$$\frac{\overset{1}{\cancel{4}}\,y^2}{\underset{25}{\cancel{100}}}-\frac{\overset{1}{\cancel{25}}\,(x-2)^2}{\underset{4}{\cancel{100}}}=\frac{100}{100}$$

$$\frac{y^2}{25}-\frac{(x-2)^2}{4}=1$$

Compare $\frac{y^2}{25}-\frac{(x-2)^2}{4}=1$ with $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$.

$\therefore\ h=2,\ k=0,\ a=\sqrt{25}=5,$ and $b=\sqrt{4}=2$

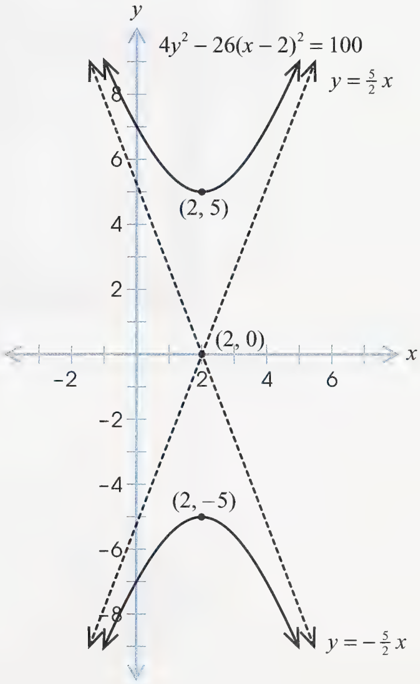
The centre of the hyperbola is at (2, 0).

The transverse axis is parallel to the y-axis, and the hyperbola opens upward and downward from the vertices.

Because $a=5$, the vertices are 5 units up and down from the centre, (2, 0). Therefore, the vertices are located at (2, –5) and (2, 5).

The slopes of the asymptotes are $\pm\frac{a}{b}=\pm\frac{5}{2}$.

Sketch the graph by first plotting the centre and the vertices. Then draw the asymptotes from their slopes. Finally, sketch the branches of the hyperbola opening upward and downward from the vertices and between the asymptotes.



To graph the asymptotes using your graphing calculator, first find their equations.

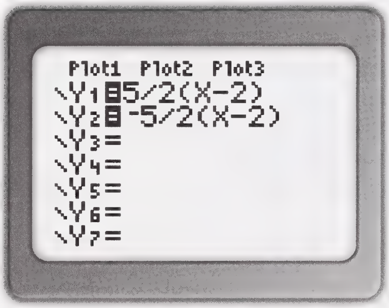
For the asymptote with slope $\frac{5}{2}$, let $m=\frac{5}{2}$ and $(x_1, y_1) = (2, 0)$.

$$\begin{aligned} \therefore y-y_1 &= m(x-x_1) \\ y-0 &= \frac{5}{2}(x-2) \\ y &= \frac{5}{2}(x-2) \end{aligned}$$

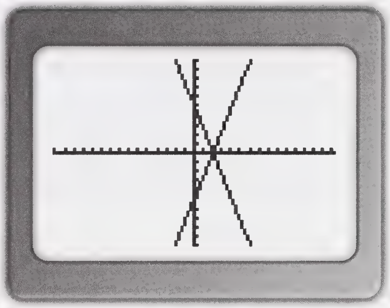
For the asymptote with slope $-\frac{5}{2}$, let $m=-\frac{5}{2}$ and $(x_1, y_1) = (2, 0)$.

$$\begin{aligned} \therefore y-y_1 &= m(x-x_1) \\ y-0 &= -\frac{5}{2}(x-2) \\ y &= -\frac{5}{2}(x-2) \end{aligned}$$

Enter $y=\frac{5}{2}(x-2)$ as Y_1 and $y=-\frac{5}{2}(x-2)$ as Y_2.



```
Plot1 Plot2 Plot3
\Y1■5/2(X-2)
\Y2■-5/2(X-2)
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



To graph the hyperbola, first solve for y.

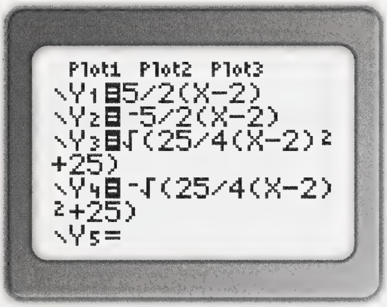
$$4y^2 - 25(x-2)^2 = 100$$

$$4y^2 = 25(x-2)^2 + 100$$

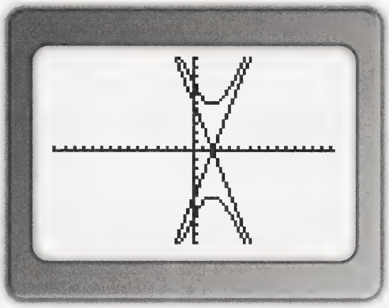
$$y^2 = \frac{25}{4}(x-2)^2 + 25$$

$$y = \pm\sqrt{\frac{25}{4}(x-2)^2 + 25}$$

Enter $y = \sqrt{\frac{25}{4}(x-2)^2 + 25}$ as Y_3 and $y = -\sqrt{\frac{25}{4}(x-2)^2 + 25}$ as Y_4.



```
Plot1 Plot2 Plot3
\Y1=5/2(X-2)
\Y2=-5/2(X-2)
\Y3=√(25/4(X-2)²
+25)
\Y4=-√(25/4(X-2)
²+25)
\Y5=
```



b. Textbook question 38 of "Applications and Problem Solving," p. 161

38.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$b^2(x-h)^2 - a^2(y-k)^2 = a^2b^2$$

$$b^2(x^2 - 2hx + h^2) - a^2(y^2 - 2ky + k^2) = a^2b^2$$

$$b^2x^2 - 2b^2hx + b^2h^2 - a^2y^2 + 2a^2ky - a^2k^2 = a^2b^2$$

$$b^2x^2 - a^2y^2 - 2b^2hx + 2a^2ky + b^2h^2 - a^2k^2 - a^2b^2 = 0$$

Compare this equation with $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

$\therefore A = b^2, C = -a^2, D = -2b^2h, E = 2a^2k$, and $F = b^2h^2 - a^2k^2 - a^2b^2$

Section 2: Activity 1 (continued)

4. a. Textbook questions 14, 16, 19, and 23 of "Practice," pp. 160 and 161

14. The centre is at (0, 0). So, $h = 0$ and $k = 0$.

The vertices are at (0, 6) and (0, −6), 6 units from the centre. Therefore, $a = 6$.

The slopes of the asymptotes, $\pm\frac{a}{b}$, are $\pm\frac{6}{4}$.

$$\begin{aligned} \therefore\ \frac{a}{b} &= \frac{6}{4} \\ \frac{6}{b} &= \frac{6}{4} \\ b &= 4 \end{aligned}$$

Substitute $a = 6$ and $b = 4$ into the standard form of a hyperbola that opens upward and downward.

$$\begin{aligned} \frac{y^2}{a^2} - \frac{x^2}{b^2} &= 1 \\ \frac{y^2}{6^2} - \frac{x^2}{4^2} &= 1 \\ \frac{y^2}{36} - \frac{x^2}{16} &= 1 \quad \leftarrow \text{standard form} \\ 16y^2 - 36x^2 &= 16(36) \\ 16y^2 - 36x^2 &= 576 \\ 0 &= 36x^2 - 16y^2 + 576 \\ 36x^2 - 16y^2 + 576 &= 0 \\ 9x^2 - 4y^2 + 144 &= 0 \quad \leftarrow \text{general form} \end{aligned}$$

16. The centre is at (–5, 1). So, $h = -5$ and $k = 1$.

The vertices are at (–8, 1) and (–2, 1), 3 units from the centre. Therefore, $a = 3$.

The slopes of the asymptotes, $\pm\frac{b}{a}$, are $\pm\frac{2}{3}$.

$$\begin{aligned} \therefore\ \frac{b}{a} &= \frac{2}{3} \\ \frac{b}{3} &= \frac{2}{3} \\ b &= 2 \end{aligned}$$

Substitute $h = -5$, $k = 1$, $a = 3$, and $b = 2$ into the standard form of a hyperbola that opens to the left and to the right.

$$\begin{aligned} \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} &= 1 \\ \frac{\left[x-(-5)\right]^2}{3^2} - \frac{(y-1)^2}{2^2} &= 1 \\ \frac{(x+5)^2}{9} - \frac{(y-1)^2}{4} &= 1 \quad \leftarrow \text{standard from} \\ 4(x+5)^2 - 9(y-1)^2 &= 9(4) \\ 4\left(x^2+10x+25\right) - 9\left(y^2-2y+1\right) &= 36 \\ 4x^2+40x+100-9y^2+18y-9 &= 36 \\ 4x^2-9y^2+40x+18y+91 &= 36 \\ 4x^2-9y^2+40x+18y+55 &= 0 \quad \leftarrow \text{general form} \end{aligned}$$

Section 2: Activity 1 (continued)

19. The centre is at (4, 0). So, $h = 4$ and $k = 0$.

The vertices are 2 units from the centre. Therefore, $a = 2$.

The slopes of the asymptotes, $\pm\frac{b}{a}$, are $\pm\frac{3}{2}$.

$$\begin{aligned}\therefore\ \frac{b}{a} &= \frac{3}{2}\\ \frac{b}{2} &= \frac{3}{2}\\ b &= 3\end{aligned}$$

Substitute $h = 4$, $k = 0$, $a = 2$, and $b = 3$ into the standard form of a hyperbola that opens to the left and to the right.

$$\begin{aligned}\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}&=1\\ \frac{(x-4)^2}{2^2}-\frac{(y-0)^2}{3^2}&=1\\ \frac{(x-4)^2}{4}-\frac{y^2}{9}&=1 \quad \leftarrow \text{standard form}\\ 9(x-4)^2-4y^2&=4(9)\\ 9(x^2-8x+16)-4y^2&=36\\ 9x^2-72x+144-4y^2&=36\\ 9x^2-4y^2-72x+108&=0 \quad \leftarrow \text{general form}\end{aligned}$$

23. The centre is at (6, –8). So, $h = 6$ and $k = -8$.

The vertices are 12 units from the centre. Therefore, $a = 12$.

The slopes of the asymptotes, $\pm\frac{a}{b}$, are ± 3.

$$\begin{aligned} \therefore\ \frac{a}{b} &= 3 \\ \frac{12}{b} &= 3 \\ \frac{12}{3} &= b \\ b &= \frac{12}{3} \\ &= 4 \end{aligned}$$

Substitute $h = 6$, $k = -8$, $a = 12$, and $b = 4$ into the standard form of a hyperbola that opens upward and downward.

$$\begin{aligned} \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} &= 1 \\ \frac{[y-(-8)]^2}{12^2} - \frac{(x-6)^2}{4^2} &= 1 \\ \frac{(y+8)^2}{144} - \frac{(x-6)^2}{16} &= 1 \quad \leftarrow \text{standard form} \\ (y+8)^2 - 9(x-6)^2 &= 144 \\ y^2 + 16y + 64 - 9(x^2 - 12x + 36) &= 144 \\ y^2 + 16y + 64 - 9x^2 + 108x - 324 &= 144 \\ -9x^2 + y^2 + 108x + 16y - 260 &= 144 \\ 9x^2 - y^2 - 108x - 16y + 404 &= 0 \quad \leftarrow \text{general form} \end{aligned}$$

Section 2: Activity 1 (continued)

b. Textbook questions 44, 45, and 52 of "Applications and Problem Solving," pp. 162 and 163

44. Because this hyperbola opens upward and downward, its graph can be obtained by transforming the graph of $y^2 - x^2 = 1$.

The centre of this hyperbola is at (4, 6), and its vertices are at (4, 5) and (4, 7). Because the vertices are 1 unit from the centre, the graph of $y^2 - x^2 = 1$ has not been stretched vertically.

The slopes of the asymptotes are $\pm\frac{1}{2}$.

$$\begin{aligned}\therefore\ \pm\frac{1\times\text{vertical stretch}}{1\times\text{horizontal stretch}} &= \pm\frac{1}{2}\\ \text{horizontal stretch} &= 2\end{aligned}$$

Therefore, the graph of $y^2 - x^2 = 1$ has been stretched horizontally by a factor of 2.

Because the centre of this hyperbola is (4, 6), the graph of $y^2 - x^2 = 1$ has been translated 4 units to the right and 6 units up.

To determine the equation of this hyperbola, first stretch $y^2 - x^2 = 1$ horizontally by a factor of 2 by substituting $\frac{x}{2}$ for x into the standard form of a hyperbola, centred at the origin, opening upward and downward.

$$\begin{aligned}y^2 - x^2 &= 1\\ y^2 - \left(\frac{x}{2}\right)^2 &= 1\\ y^2 - \frac{x^2}{4} &= 1\end{aligned}$$

Next, use the horizontal and vertical translations. Substitute $(x - 4)$ for x and $(y - 6)$ for y.

$$(y-6)^2 - \frac{(x-4)^2}{4} = 1$$

The equation of the hyperbola in standard form is $(y-6)^2 - \frac{(x-4)^2}{4} = 1$.

45. Because this hyperbola opens to the left and to the right, its graph can be obtained by transforming the graph of $x^2 - y^2 = 1$.

The centre of this hyperbola is at (4, −6), and its vertices are at (3, −6) and (5, −6). Because the vertices are 1 unit from the centre, the graph of $x^2 - y^2 = 1$ has not been stretched horizontally.

The slopes of the asymptotes are ± 1. Therefore, the graph of $x^2 - y^2 = 1$ has not been stretched vertically.

Therefore, the graph of $x^2 - y^2 = 1$ has only been translated 4 units to the right and 6 units down.

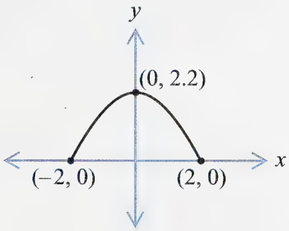
Use these translations to determine the equation. Substitute $(x - 4)$ for x and $(y + 6)$ for y into the standard form of a hyperbola, centred at the origin, opening to the left and to the right.

$$\begin{aligned} x^2 - y^2 &= 1 \\ (x-4)^2 - (y+6)^2 &= 1 \end{aligned}$$

The equation of this hyperbola in standard form is $(x - 4)^2 - (y + 6)^2 = 1$.

52. **a.** Answers may vary. A sample answer is given.

Centre the arch at (0, 2.2).



The x-intercepts are ± 2.

Section 2: Activity 1 (continued)

This hyperbola opens upward and downward; therefore, it is of the form

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

The centre, (h, k), of this curve lies along the y-axis. Therefore, $h = 0$, making the equation reduce to

$$\frac{(y-k)^2}{a^2}-\frac{x^2}{b^2}=1$$

Now, you must determine a, b, and k. Because the graph passes through $(\pm 2, 0)$, substitute $x = \pm 2$ and $y = 0$ into the equation.

$$\frac{(0-k)^2}{a^2}-\frac{(\pm 2)^2}{b^2}=1$$

$$\frac{(-k)^2}{a^2}-\frac{4}{b^2}=1$$

$$\frac{k^2}{a^2}-\frac{4}{b^2}=1 \quad ①$$

The graph also passes through $(0, 2.2)$. Substitute $x = 0$ and $y = 2.2$ into the equation.

$$\frac{(2.2-k)^2}{a^2}-\frac{0^2}{b^2}=1$$

$$\frac{(2.2-k)^2}{a^2}=1$$

$$a^2=(2.2-k)^2 \quad ②$$

Substitute $(2.2 - k)^2$ for a^2 into ①.

$$\frac{k^2}{(2.2-k)^2}-\frac{4}{b^2}=1$$

Because there are two unknowns in this equation, there are many possible solutions.

Recall that k is the y-coordinate of the centre of this hyperbola. Because the centre lies above $(0, 2.2)$, $k > 2.2$.

Suppose $k = 3.2$. Substitute $k = 3.2$ into ③.

$$\begin{aligned}\frac{k^2}{(2.2-k)^2} - \frac{4}{b^2} &= 1\\ \frac{(3.2)^2}{(-1)^2} - \frac{4}{b^2} &= 1\\ 3.2^2 - 1 &= \frac{4}{b^2}\\ 10.24 - 1 &= \frac{4}{b^2}\\ 9.24 &= \frac{4}{b^2}\\ b^2 &= \frac{4}{9.24}\\ &= \frac{400}{924}\\ &= \frac{100}{231}\end{aligned}$$

Next, find a^2. Substitute $k = 3.2$ into ②.

$$\begin{aligned}a^2 &= (2.2-3.2)^2\\ &= (-1)^2\\ &= 1\end{aligned}$$

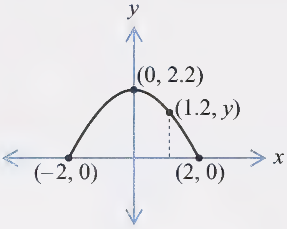
Finally, substitute $k = 3.2$, $a^2 = 1$, and $b^2 = \frac{100}{231}$ into the reduced standard form.

$$\begin{aligned}\frac{(y-k)^2}{a^2} - \frac{x^2}{b^2} &= 1\\ \frac{(y-3.2)^2}{1} - \frac{x^2}{\frac{100}{231}} &= 1\\ (y-3.2)^2 - \frac{231x^2}{100} &= 1\end{aligned}$$

Because the value of k was arbitrarily assigned a value of 3.2, this is just one of many possible equations.

Section 2: Activity 1 (continued)

b. Answers may vary.



To find the height of the arch 1.2 m from the centre, find the value of y when $x = 1.2$ using your equation from question 52.a.

$$
\begin{aligned}
(y-3.2)^2 - \frac{231x^2}{100} &= 1 \\
(y-3.2)^2 - \frac{231(1.2)^2}{100} &= 1 \\
(y-3.2)^2 &= 1 + \frac{231(1.2)^2}{100} \\
(y-3.2)^2 &= 4.3264 \\
y-3.2 &= \pm\sqrt{4.3264}
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ y-3.2 &= \sqrt{4.3264} \\
y &= \sqrt{4.3264}+3.2 \\
&= 5.28
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
y-3.2 &= -\sqrt{4.3264} \\
y &= -\sqrt{4.3264}+3.2 \\
&= 1.12
\end{aligned}
$$

Because the value of y must be less than the height of the arch at the centre, $y \doteq 1.12$. Therefore, the height of the arch 1.2 m from the centre is approximately 1.12 m.

5. Textbook questions 33 and 35 of "Practice," p. 161

33. a. Convert the equation into standard form.

$$
\begin{aligned}
x^2-4y^2-6x-8y-11&=0\\
x^2-6x-4y^2-8y&=11\\
\left(x^2-6x\right)-4\left(y^2+2y\right)&=11\\
\left[\left(x^2-6x+9\right)-9\right]-4\left[\left(y^2+2y+1\right)-1\right]&=11\\
(x-3)^2-9-4\left[(y+1)^2-1\right]&=11\\
(x-3)^2-9-4(y+1)^2+4&=11\\
(x-3)^2-4(y+1)^2&=16\\
\frac{(x-3)^2}{16}-\frac{\overset{1}{\cancel{4}}(y+1)^2}{\underset{4}{\cancel{16}}}&=\frac{16}{16}\\
\frac{(x-3)^2}{16}-\frac{(y+1)^2}{4}&=1
\end{aligned}
$$

Compare this equation with $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.

$\therefore\ h=3,\ k=-1,\ a=\sqrt{16}=4,$ and $b=\sqrt{4}=2$

The centre, (h, k), is at $(3, -1)$.

b. The curve opens to the left and to the right, and the vertices are a = 4 units from the centre. Therefore, the vertices are $(3-4, -1)=(-1, -1)$ and $(3+4, -1)=(7, -1)$.

c. The slopes of the asymptotes are

$$
\begin{aligned}
m&=\pm\frac{b}{a}\\
&=\pm\frac{2}{4}\\
&=\pm\frac{1}{2}
\end{aligned}
$$

Section 2: Activity 1 (continued)

Because the asymptotes pass through the centre, $(3, -1)$, use $y - y_1 = m(x - x_1)$ to determine their equations. Let $m = \pm\frac{1}{2}$ and $(x_1, y_1) = (3, -1)$.

When $m = \frac{1}{2}$,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y + 1 &= \frac{1}{2}(x - 3) \\ 2(y + 1) &= x - 3 \\ 2y + 2 &= x - 3 \\ 0 &= x - 2y - 5 \\ x - 2y - 5 &= 0 \end{aligned}$$

When $m = -\frac{1}{2}$,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y + 1 &= -\frac{1}{2}(x - 3) \\ -2(y + 1) &= x - 3 \\ -2y - 2 &= x - 3 \\ 0 &= x + 2y - 1 \\ x + 2y - 1 &= 0 \end{aligned}$$

The equations of the asymptotes are $x - 2y - 5 = 0$ and $x + 2y - 1 = 0$.

35. **a.** Convert the equation into standard form.

$$\begin{aligned} x^2 - 10x - 36y^2 + 216y &= 335 \\ (x^2 - 10x) - 36(y^2 - 6y) &= 335 \\ \left[(x^2 - 10x + 25) - 25\right] - 36\left[(y^2 - 6y + 9) - 9\right] &= 335 \\ (x - 5)^2 - 25 - 36\left[(y - 3)^2 - 9\right] &= 335 \\ (x - 5)^2 - 25 - 36(y - 3)^2 + 324 &= 335 \\ (x - 5)^2 - 36(y - 3)^2 &= 36 \\ \frac{(x - 5)^2}{36} - \frac{\cancel{36}(y - 3)^2}{\cancel{36}} &= \frac{36}{36} \\ \frac{(x - 5)^2}{36} - \frac{(y - 3)^2}{1} &= 1 \end{aligned}$$

Compare this equation with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.

$\therefore\ h = 5,\ k = 3,\ a = \sqrt{36} = 6,$ and $b = \sqrt{1} = 1$

The centre, (h, k), is at $(5, 3)$.

b. The curve opens to the left and to the right, and the vertices are $a = 6$ units from the centre. Therefore, the vertices are $(5 - 6, 3) = (-1, 3)$ and $(5 + 6, 3) = (11, 3)$.

c. The slopes of the asymptotes are

$$\begin{aligned} m &= \pm\frac{b}{a} \\ &= \pm\frac{1}{6} \end{aligned}$$

Because the asymptotes pass through the centre (5, 3), use $y - y_1 = m(x - x_1)$ to determine their equations. Let $m = \pm\frac{1}{6}$ and $(x_1, y_1) = (5, 3)$.

When $m = \frac{1}{6}$,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= \frac{1}{6}(x - 5) \\ 6(y - 3) &= x - 5 \\ 6y - 18 &= x - 5 \\ 0 &= x - 6y + 13 \\ x - 6y + 13 &= 0 \end{aligned}$$

When $m = -\frac{1}{6}$,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= -\frac{1}{6}(x - 5) \\ -6(y - 3) &= x - 5 \\ -6y + 18 &= x - 5 \\ 0 &= x + 6y - 23 \\ x + 6y - 23 &= 0 \end{aligned}$$

The equations of the asymptotes are $x - 6y + 13 = 0$ and $x + 6y - 23 = 0$.

Section 2: Activity 1 (continued)

6. a.

$$\begin{aligned}
9x^2-y^2-36x-2y+26&=0\\
9x^2-36x-y^2-2y&=-26\\
9(x^2-4x)-(y^2+2y)&=-26\\
9\left[(x^2-4x+4)-4\right]-\left[(y^2+2y+1)-1\right]&=-26\\
9\left[(x-2)^2-4\right]-\left[(y+1)^2-1\right]&=-26\\
9(x-2)^2-36-(y+1)^2+1&=-26\\
9(x-2)^2-(y+1)^2&=9\\
\frac{\cancel{9}(x-2)^2}{\cancel{9}}-\frac{(y+1)^2}{9}&=\frac{9}{9}\\
\frac{(x-2)^2}{1}-\frac{(y+1)^2}{9}&=1
\end{aligned}$$

Compare this equation with $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.

$\therefore\ h=2,\ k=-1,\ a=\sqrt{1}=1,$ and $b=\sqrt{9}=3$

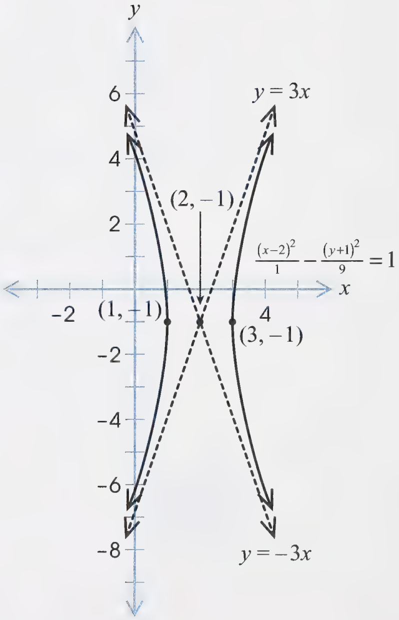
The centre, (h, k), is at $(2, -1)$.

The curve opens to the left and to the right, and the vertices are $a = 1$ unit from the centre. Therefore, the vertices are $(2-1, -1) = (1, -1)$ and $(2+1, -1) = (3, -1)$.

The slopes of the asymptotes are

$$\begin{aligned}
m&=\pm\frac{b}{a}\\
&=\pm\frac{3}{1}\\
&=\pm3
\end{aligned}$$

Graph the hyperbola.



b.

$$\begin{aligned}
x^2 - y^2 - 4x - 6y - 5 &= 0 \\
x^2 - 4x - y^2 - 6y &= 5 \\
\left(x^2 - 4x\right) - \left(y^2 + 6y\right) &= 5 \\
\left[\left(x^2 - 4x + 4\right) - 4\right] - \left[\left(y^2 + 6y + 9\right) - 9\right] &= 5 \\
(x-2)^2 - 4 - (y+3)^2 + 9 &= 5 \\
(x-2)^2 - (y+3)^2 &= 0
\end{aligned}$$

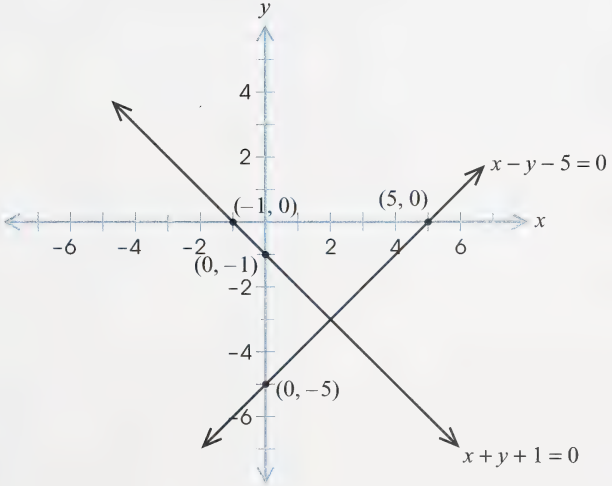
This is not a primary hyperbola.

Factor.

$$\begin{aligned}
\left[(x-2)+(y+3)\right]\left[(x-2)-(y+3)\right] &= 0 \\
(x-2+y+3)(x-2-y-3) &= 0 \\
(x+y+1)(x-y-5) &= 0
\end{aligned}$$

$\therefore x + y + 1 = 0 \quad \text{or} \quad x - y - 5 = 0$

Therefore, the original equation is a pair of intersecting lines. This is a degenerate form of the hyperbola.



Section 2: Activity 1 (continued)

7. Textbook question 4 of "Applications and Problem Solving," p. 137

4. Begin by looking for a pattern in the units digit.

$17^1 = 17$

$17^2 = 289$

$17^3 = 4913$

$17^4 = 83\,521$

$17^5 = 1\,419\,857$

$17^6 = 24\,137\,569$

$17^7 = 410\,338\,673$

$17^8 = 6\,975\,757\,441$

The units digits of the standard form of the powers of 17 have a pattern of 4 repeating digits: 7, 9, 3, and 1. To determine the units digit of 17^{133}, divide 133 by 4 and use the remainder.

$$\begin{array}{r} 33 \\ 4\overline{)133} \\ \underline{12} \\ 13 \\ \underline{12} \\ 1 \end{array}$$

Therefore, 17^{133} ends in the same digit as 17^1. The units digit is 7.

Section 2: Activity 2

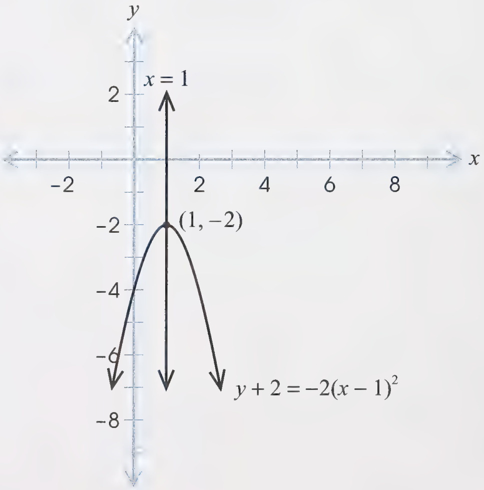
1. a. Compare $y + 2 = -2(x - 1)^2$ with $y - k = a(x - h)^2$.

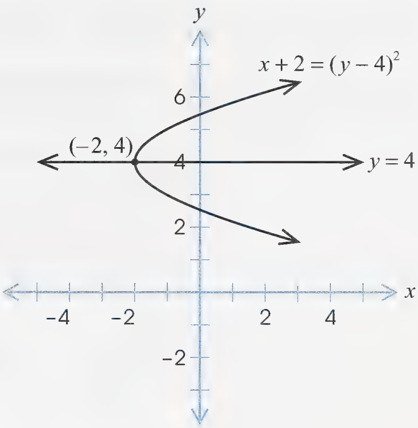
$\therefore\ a = -2,\ h = 1,\text{ and } k = -2$

The vertex, (h, k), is at $(1, -2)$ and the axis of symmetry is $x = 1$.

The graph opens downward since $a < 0$.

The domain is the set of reals, and the range is $y \leq -2$.





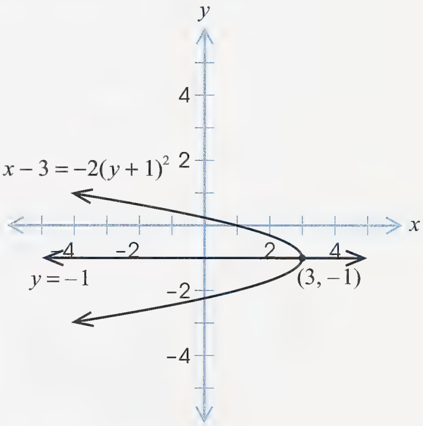
b. Compare $x + 2 = (y - 4)^2$ with $x - h = a(y - k)^2$.

$\therefore\ a = 1,\ h = -2,$ and $k = 4$

The vertex, (h, k), is at $(-2, 4)$, and the axis of symmetry is $y = 4$.

Because $a > 0$, the graph opens to the right.

The domain is $x \geq 2$, and the range is the set of reals.



c. Compare $x - 3 = -2(y + 1)^2$ with $x - h = a(y - k)^2$.

$\therefore\ a = -2,\ h = 3,$ and $k = -1$

The vertex, (h, k), is at $(3, -1)$, and the axis of symmetry is $y = -1$.

Because $a < 0$, the graph opens to the left.

The domain is $x \leq 3$, and the range is the set of reals.

Section 2: Activity 2 (continued)

2. a. Textbook questions 12 and 25 of "Practice," p. 168

12. Because the parabola opens to the right, the standard form of the equation of this parabola is $x - h = a(y - k)^2$.

The vertex is at $(-7, -5)$; so, $h = -7$ and $k = -5$.

$$\begin{aligned} \therefore\ x-h&=a(y-k)^2 \\ x-(-7)&=a\left[y-(-5)\right]^2 \\ x+7&=a(y+5)^2 \end{aligned}$$

Next, find a. Because the graph passes through $(2, -1)$, substitute $x = 2$ and $y = -1$ into the equation.

$$\begin{aligned} x+7&=a(y+5)^2 \\ 2+7&=a(-1+5)^2 \\ 9&=a(4)^2 \\ 9&=16a \\ a&=\frac{9}{16} \end{aligned}$$

The equation in standard form is $x+7=\frac{9}{16}(y+5)^2$.

Convert the equation into general form.

$$\begin{aligned} x+7&=\frac{9}{16}(y+5)^2 \\ 16(x+7)&=9(y+5)^2 \\ 16x+112&=9(y^2+10y+25) \\ 16x+112&=9y^2+90y+225 \\ 0&=9y^2-16x+90y+113 \\ 9y^2-16x+90y+113&=0 \end{aligned}$$

The equation in general form is $9y^2 - 16x + 90y + 113 = 0$.

25. Answers may vary. A sample answer is given.

Because the parabola opens to the right, the standard form of its equation is $x - h = a(y - k)^2$.

The vertex is at $(-1, -2)$; so, $h = -1$ and $k = -2$.

$$\begin{aligned} \therefore\ x-h&=a(y-k)^2 \\ x-(-1)&=a[y-(-2)]^2 \\ x+1&=a(y+2)^2 \end{aligned}$$

Next, find a. Because the graph passes through $(0, 1)$, substitute $x = 0$ and $y = 1$ into the equation.

$$\begin{aligned} x+1&=a(y+2)^2 \\ 0+1&=a(1+2)^2 \\ 1&=a(3)^2 \\ 1&=9a \\ a&=\frac{1}{9} \end{aligned}$$

The equation in standard form is $x+1=\frac{1}{9}(y+2)^2$.

Convert the equation into general form.

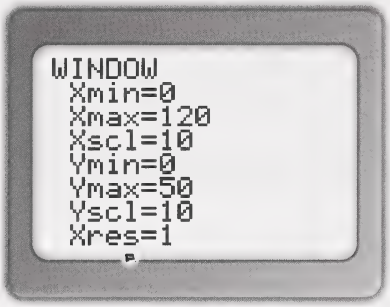
$$\begin{aligned} x+1&=\frac{1}{9}(y+2)^2 \\ 9(x+1)&=(y+2)^2 \\ 9x+9&=y^2+4y+4 \\ 0&=y^2-9x+4y-5 \\ y^2-9x+4y-5&=0 \end{aligned}$$

The general form of the equation is $y^2 - 9x + 4y - 5 = 0$.

Section 2: Activity 2 (continued)

b. **Textbook questions 47 and 49.c. of "Applications and Problem Solving," p. 169**

47. **a.** Use the following window settings.



```
WINDOW
 Xmin=0
 Xmax=120
 Xscl=10
 Ymin=0
 Ymax=50
 Yscl=10
 Xres=1
```

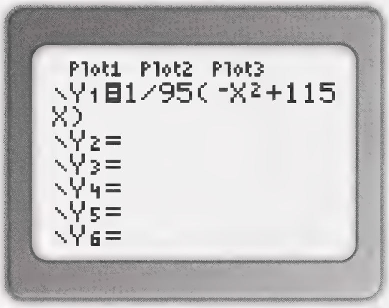
Solve the equation for y.

$$x^2 - 115x + 95y = 0$$

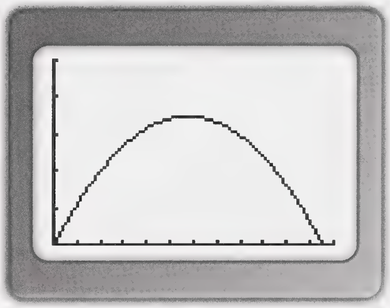
$$95y = -x^2 + 115x$$

$$y = \frac{1}{95}\left(-x^2 + 115x\right)$$

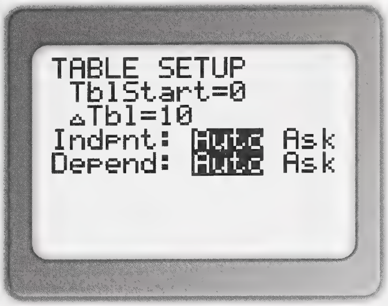
Enter this equation into your graphing calculator.



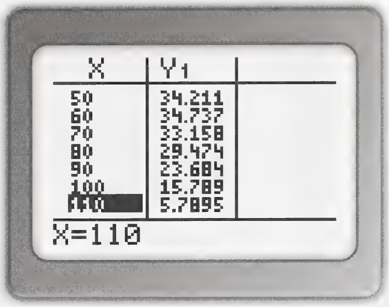
```
 Plot1 Plot2 Plot3
\Y1=1/95(-X²+115
X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



Use the Table feature to determine the ball's height 110 m from home plate.



```
TABLE SETUP
 TblStart=0
 ΔTbl=10
Indpnt: Auto Ask
Depend: Auto Ask
```

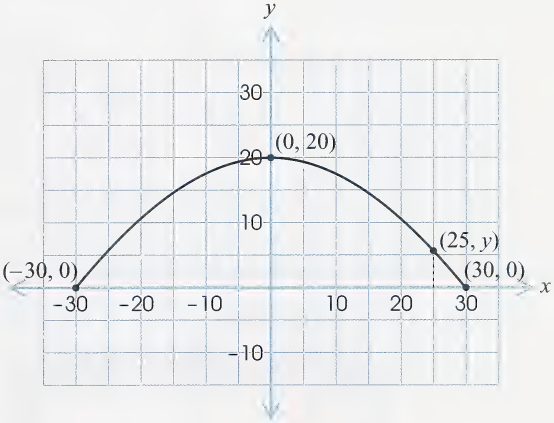


```
  X    | Y1
  50   | 34.211
  60   | 34.737
  70   | 33.158
  80   | 29.474
  90   | 23.684
  100  | 15.789
  110  | 5.7895
X=110
```

Because the ball is still approximately 5.8 m above the ground, it will clear the 2-m wall.

b. A player could not jump to reach a height of 5.8 m to catch the ball.

49. c. Position the vertex at (0, 20). Therefore, the arch meets the ground at (±30, 0), since the arch spans a distance of 60 m.



Determine the equation of the parabola. Because the parabola opens downward, it is of the form $y - k = a(x - h)^2$.

The vertex is at (0, 20); so, $h = 0$ and $k = 20$.

$$\therefore\ y-k=a(x-h)^2$$
$$y-20=a(x-0)^2$$
$$y-20=ax^2$$

Section 2: Activity 2 (continued)

Next, find a. Because the graph passes through (30, 0), substitute $x = 30$ and $y = 0$ into the equation.

$$\begin{aligned} y-20&=ax^2\\ 0-20&=a(30)^2\\ -20&=900a\\ a&=-\frac{20}{900}\\ &=-\frac{1}{45} \end{aligned}$$

The equation of the parabola is $y-20=-\frac{1}{45}x^2$ or $y=-\frac{1}{45}x^2+20$.

Now, determine the height at $x = 25$.

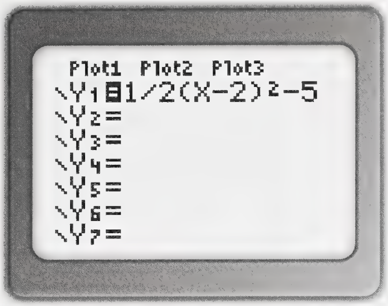
$$\begin{aligned} y&=-\frac{1}{45}x^2+20\\ &=-\frac{1}{45}(25)^2+20\\ &\doteq 6.1 \end{aligned}$$

The height of the arch 25 m from the axis of symmetry of the parabola is about 6.1 m.

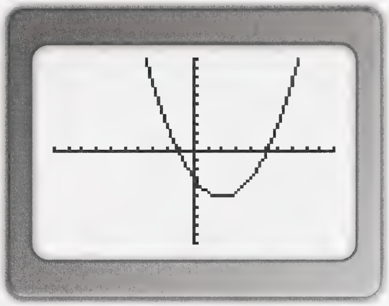
3. **Textbook questions 36, 37, and 39 of "Practice," p. 168**

36.
$$\begin{aligned} x^2-4x-2y-6&=0\\ x^2-4x&=2y+6\\ 2y+6&=x^2-4x\\ 2y+6&=(x^2-4x+4)-4\\ 2y+6&=(x-2)^2-4\\ 2y+10&=(x-2)^2\\ 2(y+5)&=(x-2)^2\\ y+5&=\frac{1}{2}(x-2)^2 \quad \leftarrow \text{standard form} \end{aligned}$$

The graph opens upward from vertex (2, –5).



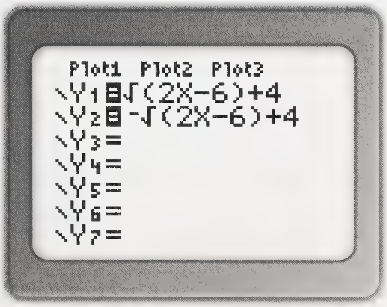
```
 Plot1  Plot2  Plot3
\Y1=1/2(X-2)²-5
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



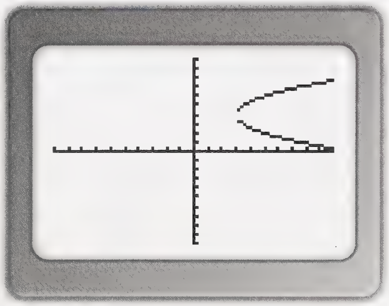
37. $y^2 - 2x - 8y + 22 = 0$

$$\begin{aligned}
y^2 - 8y &= 2x - 22 \\
2x - 22 &= y^2 - 8y \\
2x - 22 &= \left(y^2 - 8y + 16\right) - 16 \\
2x - 22 &= (y-4)^2 - 16 \\
2x - 6 &= (y-4)^2 \\
2(x-3) &= (y-4)^2 \\
x - 3 &= \frac{1}{2}(y-4)^2 \quad \leftarrow \text{standard form}
\end{aligned}$$

The graph opens to the right from vertex (3, 4).



```
 Plot1  Plot2  Plot3
\Y1=√(2X-6)+4
\Y2=-√(2X-6)+4
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

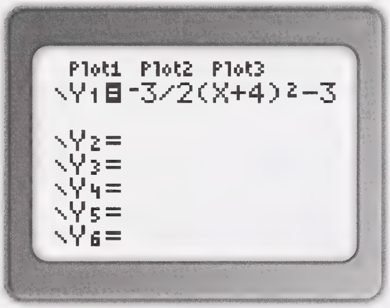


Section 2: Activity 2 (continued)

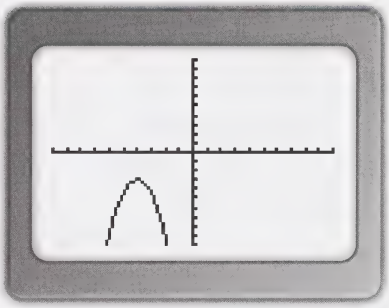
39. $3x^2+24x+2y+54=0$

$$\begin{aligned}
2y+54&=-3x^2-24x\\
2y+54&=-3\left(x^2+8x\right)\\
2y+54&=-3\left[\left(x^2+8x+16\right)-16\right]\\
2y+54&=-3\left[(x+4)^2-16\right]\\
2y+54&=-3(x+4)^2+48\\
2y+6&=-3(x+4)^2\\
2(y+3)&=-3(x+4)^2\\
y+3&=-\frac{3}{2}(x+4)^2 \quad \leftarrow \text{standard form}
\end{aligned}$$

This graph opens downward from vertex (–4, –3).



```
Plot1 Plot2 Plot3
\Y1=-3/2(X+4)²-3
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



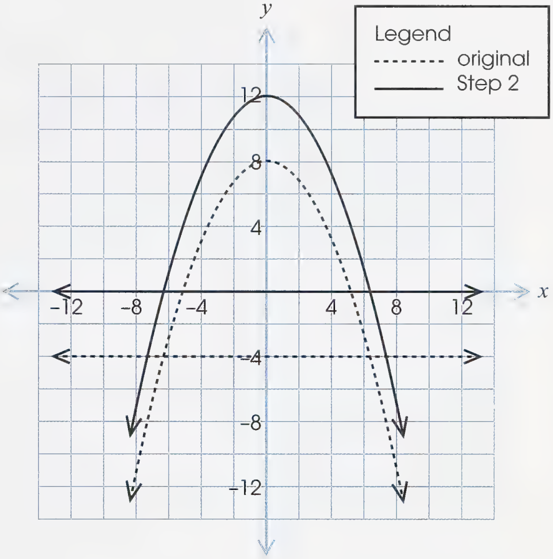
4. To stretch the parabola vertically by a factor of $\frac{1}{3}$ about the line $x = -4$, follow these steps:

Step 1: Solve the equation for y.

$$3x^2 + 10y - 80 = 0$$
$$10y = -3x^2 + 80$$
$$y = -\frac{3}{10}x^2 + 8$$

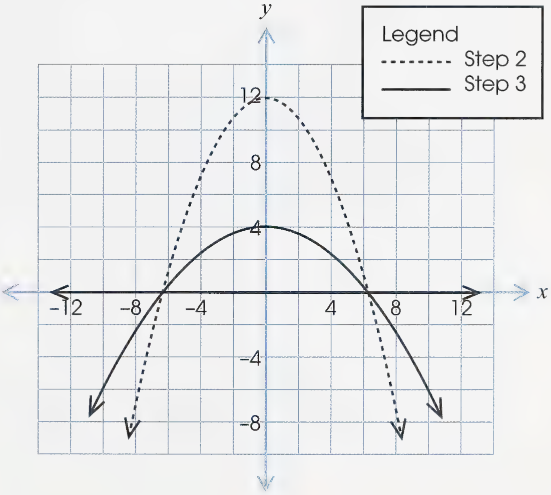
Step 2: Move the graph 4 units up so the graph is stretched about the x-axis.

$$y = -\frac{3}{10}x^2 + 8 + 4$$
$$y = -\frac{3}{10}x^2 + 12$$



Step 3: Stretch the graph vertically by a factor of $\frac{1}{3}$ about the x-axis. Multiply y by 3.

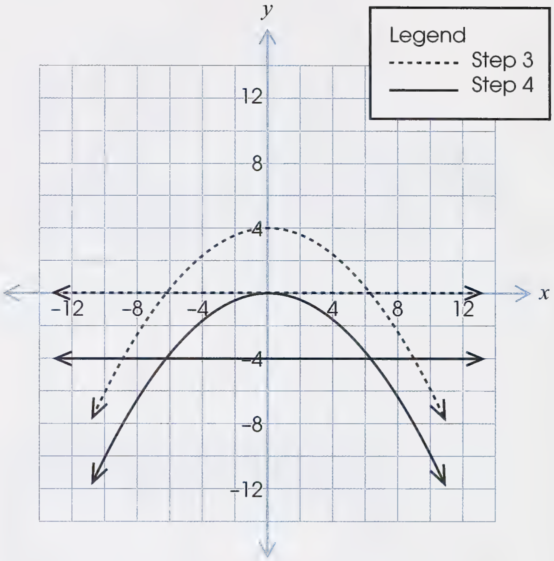
$$3y = -\frac{3}{10}x^2 + 12$$
$$y = -\frac{1}{10}x^2 + 4$$



Section 2: Activity 2 (continued)

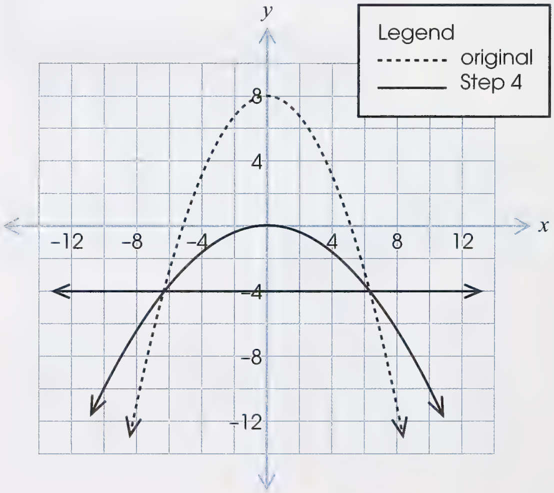
Step 4: Move the graph 4 units down so it is in its original place.

$$y = -\frac{1}{10}x^2 + 4 - 4$$
$$y = -\frac{1}{10}x^2$$



Step 5: Convert the equation into general form.

$$y = -\frac{1}{10}x^2$$
$$10y = -x^2$$
$$x^2 + 10y = 0$$



5. Textbook question 9 of "Applications and Problem Solving," p. 173

9. Answers may vary. A sample answer is given.

Work Backward	Solution
BREAD	FLOUR
BROAD	FLOOR
BROOD	FLOOD
BLOOD	BLOOD
FLOOD	BROOD
FLOOR	BROAD
FLOUR	BREAD

Section 2: Follow-up Activities

Extra Help

1. Compare $\frac{(x+1)^2}{4}-\frac{(y-2)^2}{9}=1$ with $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}$.

$\therefore\ h=-1,\ k=2,\ a=\sqrt{4}=2,$ and $b=\sqrt{9}=3$

The centre, (h, k), of the hyperbola is at $(-1, 2)$.

Because the hyperbola opens to the right and to the left, the vertices lie $a = 2$ units to the right and to the left of centre. Thus, the vertices are

$(-1-2, 2) = (-3, 2)$ and $(-1+2, 2) = (1, 2)$

Therefore, the domain is $x \leq -3$ or $x \geq 1$, and the range is the set of reals.

Convert the equation into general form.

$$\frac{(x+1)^2}{4}-\frac{(y-2)^2}{9}=1$$

$$\overset{9}{\cancel{36}}\left[\frac{(x+1)^2}{\underset{1}{\cancel{4}}}\right]-\overset{4}{\cancel{36}}\left[\frac{(y-2)^2}{\underset{1}{\cancel{9}}}\right]=36(1)$$ ← Multiply each term by the LCD, 36.

$$9(x+1)^2-4(y-2)^2=36$$

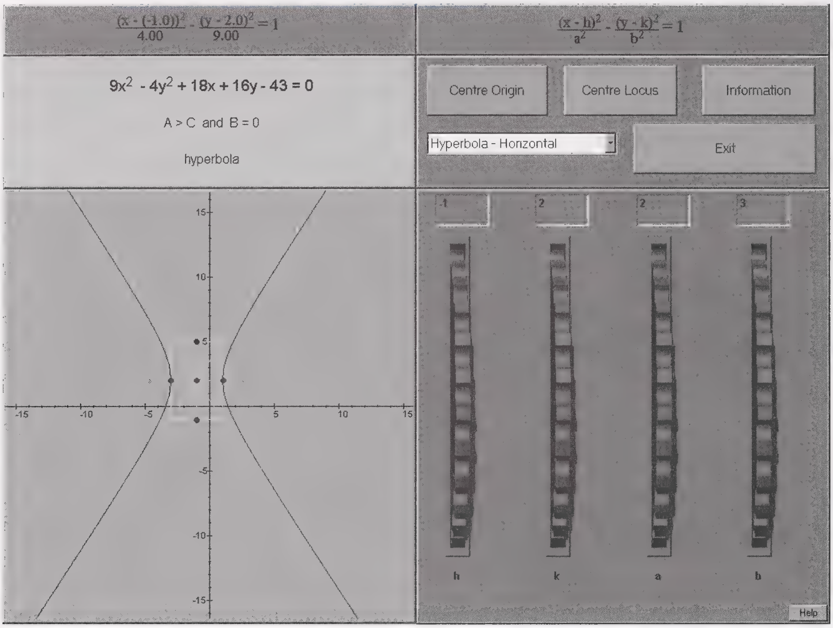
$$9\left(x^2+2x+1\right)-4\left(y^2-4y+4\right)=36$$

$$9x^2+18x+9-4y^2+16y-16=36$$

$$9x^2-4y^2+18x+16y-43=0$$ ← general form

Extra Help (continued)

Check



$\frac{(x-(-1.0))^2}{4.00} - \frac{(y-2.0)^2}{9.00} = 1$

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$9x^2 - 4y^2 + 18x + 16y - 43 = 0$

A > C and B = 0

hyperbola

Hyperbola - Horizontal

h: -1

k: 2

a: 2

b: 3

2.

$$16x^2 - 9y^2 - 32x - 72y + 16 = 0$$

$$16x^2 - 32x - 9y^2 - 72y = -16 \quad \leftarrow \text{Group the terms.}$$

$$16(x^2 - 2x) - 9(y^2 + 8y) = -16 \quad \leftarrow \text{Remove the coefficients from } x^2 \text{ and } y^2.$$

$$16\left[(x^2 - 2x + 1) - 1\right] - 9\left[(y^2 + 8y + 16) - 16\right] = -16 \quad \leftarrow \text{Complete the square.}$$

$$16\left[(x-1)^2 - 1\right] - 9\left[(y+4)^2 - 16\right] = -16$$

$$16(x-1)^2 - 16 - 9(y+4)^2 + 144 = -16$$

$$16(x-1)^2 - 9(y+4)^2 = -144$$

$$\frac{\overset{-1}{\cancel{16}}(x-1)^2}{-\underset{9}{\cancel{144}}} - \frac{\overset{-1}{\cancel{9}}(y+4)^2}{-\underset{16}{\cancel{144}}} = \frac{-144}{-144}$$

$$-\frac{(x-1)^2}{9} + \frac{(y+4)^2}{16} = 1$$

$$\frac{(y+4)^2}{16} - \frac{(x-1)^2}{9} = 1$$

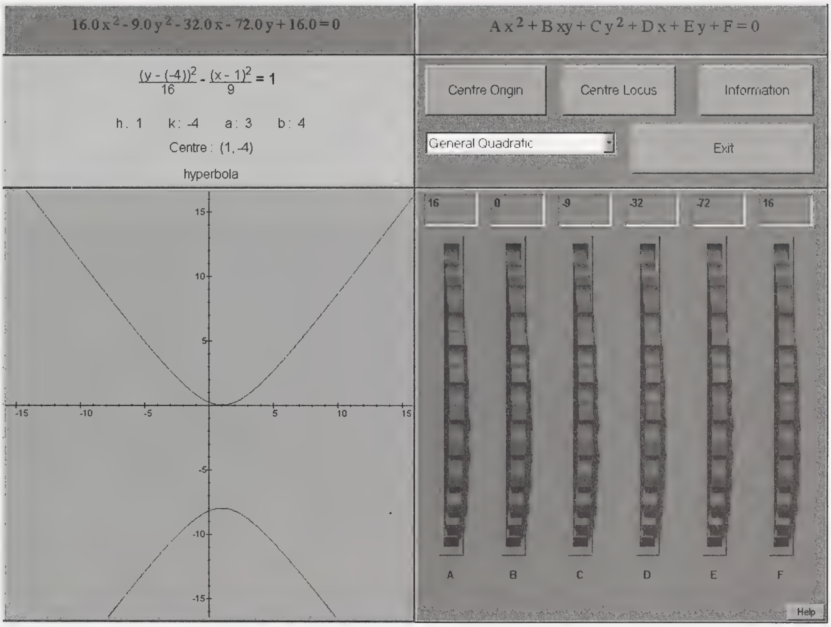
The centre of this hyperbola is at (1, −4).

Because the hyperbola opens upward and downward, the vertices lie $a = \sqrt{16} = 4$ units above and below the centre. Therefore, the vertices are

$$(1, -4 - 4) = (1, -8) \text{ and } (1, -4 + 4) = (1, 0)$$

The domain is the set of reals, and the range is $y \leq -8$ or $y \geq 0$.

Check



$$16.0x^2 - 9.0y^2 - 32.0x - 72.0y + 16.0 = 0$$

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$$\frac{(y-(-4))^2}{16} - \frac{(x-1)^2}{9} = 1$$

h. 1 k: -4 a: 3 b: 4

Centre : (1, -4)

hyperbola

General Quadratic

A: 16
B: 0
C: -9
D: -32
E: -72
F: 16

Extra Help (continued)

3. The centre, (h, k), is at $(2, 3)$; so, $h = 2$ and $k = 3$.

Because the transverse axis is horizontal, the hyperbola opens to the left and to the right.

Because the transverse axis is 6 units in length,

$$\begin{aligned} 2a &= 6 \\ a &= 3 \end{aligned}$$

The vertices of this hyperbola are

$$(2-3, 3) = (-1, 3) \text{ and } (2+3, 3) = (5, 3)$$

The slopes of the asymptotes are $\pm\frac{1}{2}$.

$$\begin{aligned} \therefore\ \pm\frac{b}{a} &= \pm\frac{1}{2} \\ \frac{b}{3} &= \frac{1}{2} \\ 2b &= 3 \\ b &= \frac{3}{2} \end{aligned}$$

Substitute $h = 2$, $k = 3$, $a = 3$, and $b = \frac{3}{2}$ into the standard form of a hyperbola that opens to the left and to the right.

$$\begin{aligned} \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} &= 1 \\ \frac{(x-2)^2}{3^2} - \frac{(y-3)^2}{\left(\frac{3}{2}\right)^2} &= 1 \\ \frac{(x-2)^2}{9} - \frac{(y-3)^2}{\frac{9}{4}} &= 1 \quad \leftarrow \text{standard form} \end{aligned}$$

Convert the equation into general form.

$$\frac{(x-2)^2}{9}-\frac{(y-3)^2}{\frac{9}{4}}=1$$

$$\frac{(x-2)^2}{9}-\frac{4(y-3)^2}{9}=1$$

$$\cancel{9}\left[\frac{(x-2)^2}{\cancel{9}}\right]-\cancel{9}\left[\frac{4(y-3)^2}{\cancel{9}}\right]=9(1)$$

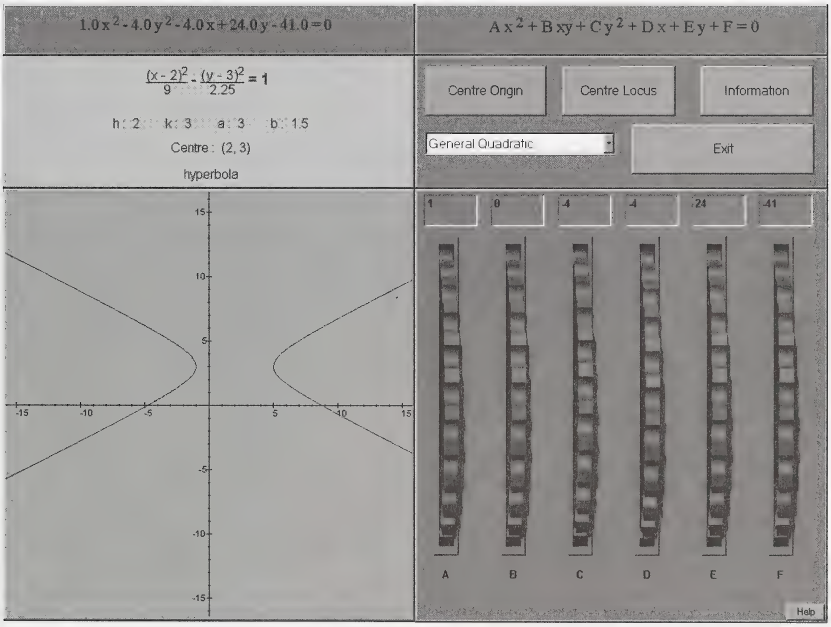
$$(x-2)^2-4(y-3)^2=9$$

$$x^2-4x+4-4(y^2-6y+9)=9$$

$$x^2-4x+4-4y^2+24y-36=9$$

$$x^2-4y^2-4x+24y-41=0 \quad \leftarrow \text{general form}$$

Check



$1.0x^2-4.0y^2-4.0x+24.0y-41.0=0$

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

$\frac{(x-2)^2}{9}-\frac{(y-3)^2}{2.25}=1$

h: 2 k: 3 a: 3 b: 1.5

Centre : (2, 3)

hyperbola

General Quadratic

A: 1 B: 0 C: -4 D: -4 E: 24 F: -41

Enrichment

1. Let $P(x, y)$ be any point on the hyperbola; let $(0, 5)$ be focus F_1; and let $(0, -5)$ be focus F_2.

$$\therefore \left|PF_1 - PF_2\right| = 8$$

$$\left|\sqrt{(x-0)^2+(y-5)^2} - \sqrt{(x-0)^2+\left[y-(-5)\right]^2}\right| = 8$$

$$\sqrt{x^2+(y-5)^2} - \sqrt{x^2+(y+5)^2} = \pm 8$$

$$\sqrt{x^2+(y-5)^2} = \pm 8 + \sqrt{x^2+(y+5)^2}$$

$$\left(\sqrt{x^2+(y-5)^2}\right)^2 = \left(\pm 8 + \sqrt{x^2+(y+5)^2}\right)^2$$

$$\cancel{x}^2+(y-5)^2 = 64 \pm 16\sqrt{x^2+(y+5)^2} + \cancel{x}^2+(y+5)^2$$

$$(y-5)^2 = 64 \pm 16\sqrt{x^2(y+5)^2} + (y+5)^2$$

$$\cancel{y}^2 - 10y + \cancel{25} = 64 \pm 16\sqrt{x^2+(y+5)^2} + \cancel{y}^2 + 10y + \cancel{25}$$

$$-10y = 64 \pm 16\sqrt{x^2+(y+5)^2} + 10y$$

$$\pm 16\sqrt{x^2+(y+5)^2} = 64 + 20y$$

$$\pm 4\sqrt{x^2+(y+5)^2} = 16 + 5y \quad \leftarrow \text{Divide each term by the LCM, 4.}$$

$$\left(\pm 4\sqrt{x^2+(y+5)^2}\right)^2 = (16+5y)^2$$

$$16\left[x^2+(y+5)^2\right] = 256 + 160y + 25y^2$$

$$16\left(x^2+y^2+10y+25\right) = 256 + 160y + 25y^2$$

$$16x^2 + 16y^2 + \cancel{160y} + 400 = 256 + \cancel{160y} + 25y^2$$

$$16x^2 + 16y^2 + 400 = 256 + 25y^2$$

$$144 = 9y^2 - 16x^2$$

$$9y^2 - 16x^2 = 144$$

$$\frac{\overset{1}{\cancel{9}}y^2}{\underset{16}{\cancel{144}}} - \frac{\overset{1}{\cancel{16}}x^2}{\underset{9}{\cancel{144}}} = \frac{144}{144}$$

$$\frac{y^2}{16} - \frac{x^2}{9} = 1$$

2. Compare $\frac{x^2}{144}-\frac{y^2}{25}=1$ with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

$\therefore\ a^2 = 144$ and $b^2 = 25$

For the hyperbola,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ c^2 &= 144 + 25 \\ c^2 &= 169 \\ c &= 13 \end{aligned}$$

The hyperbola opens to the left and to the right. Therefore, the foci are $(\pm c, 0) = (\pm 13, 0)$.

Credits

All images in this module were created by or for Alberta Education with the following noted exceptions:

Welcome Page
Image Club Studiogear/Getty Images

Page
2–3 **collage:** Photodisc/Getty Images
4 Photodisc/Getty Images
8 Image Club Studiogear/Getty Images
10 Photodisc/Getty Images
14 Image Club Studiogear/Getty Images
15 **bottom:** Photodisc/Getty Images
16 NOAO/AURA/NSF
19 Image Club Studiogear/Getty Images
20 Image Club Studiogear/Getty Images
21 **bottom:** Image Club Studiogear/Getty Images
24 © 2005–2006 www.clipart.com
26 Courtesy NASA/JPL-Caltech
27 **bottom:** Image Club Studiogear/Getty Images
29 © 2005–2006 www.clipart.com
32 Image Club Studiogear/Getty Images
33 Image Club Studiogear/Getty Images
37 Photodisc/Getty Images
40 **top:** Image Club Studiogear/Getty Images
42 Image Club Studiogear/Getty Images
43 © Galina Barskaya/shutterstock.com
44 **top:** Dimitri Vervits/Digital Vision/Getty Images
bottom: © 2005–2006 www.clipart.com
45 **top:** © 2005–2006 www.clipart.com
bottom: Photodisc/Getty Images
47 **bottom:** Eyewire/Getty Images
49 **bottom:** Image Club Studiogear/Getty Images
52 Eyewire/Getty Images
53 Photodisc/Getty Images
66 Image Club Studiogear/Getty Images
69 **top:** © Yuri Arcurs/shutterstock.com
71 Photodisc/Getty Images
74 Image Club Studiogear/Getty Images
78 **top:** Photodisc/Getty Images
81 Photodisc/Getty Images
84 **bottom:** Image Club Studiogear/Getty Images
87 Eyewire/Getty Images
88 **left:** © Cathleen Clapper/shutterstock.com
89 Photodisc/Getty Images

© 2006